

$$\begin{bmatrix} L & i & n & e & a & r \\ A & l & g & e & b & r & a \\ R & e & f & r & e & s & h & e & r \end{bmatrix}$$

EECS 367

Intro. to Autonomous Robotics

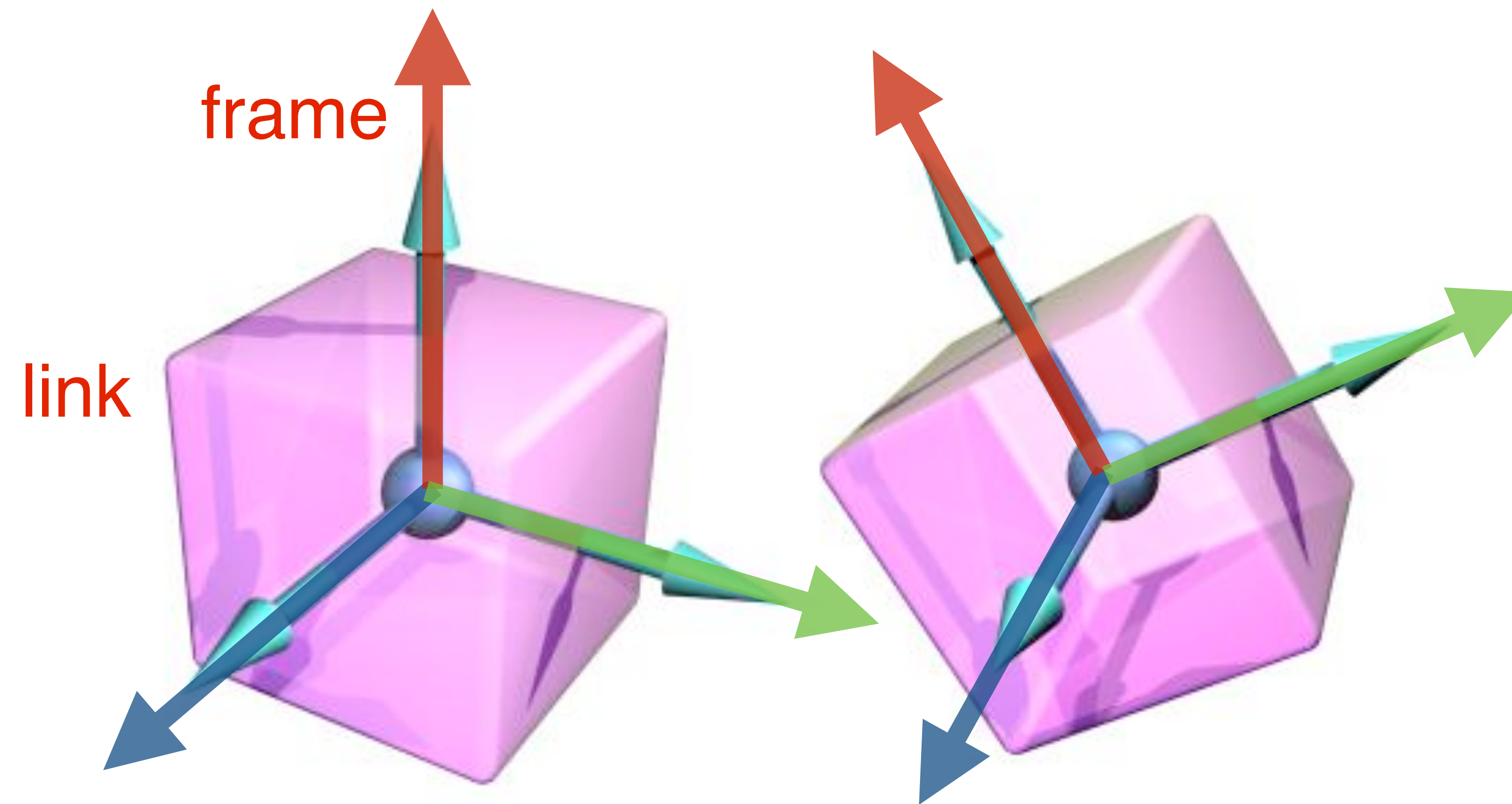
ROB 320

Robot Operating Systems

Winter 2022

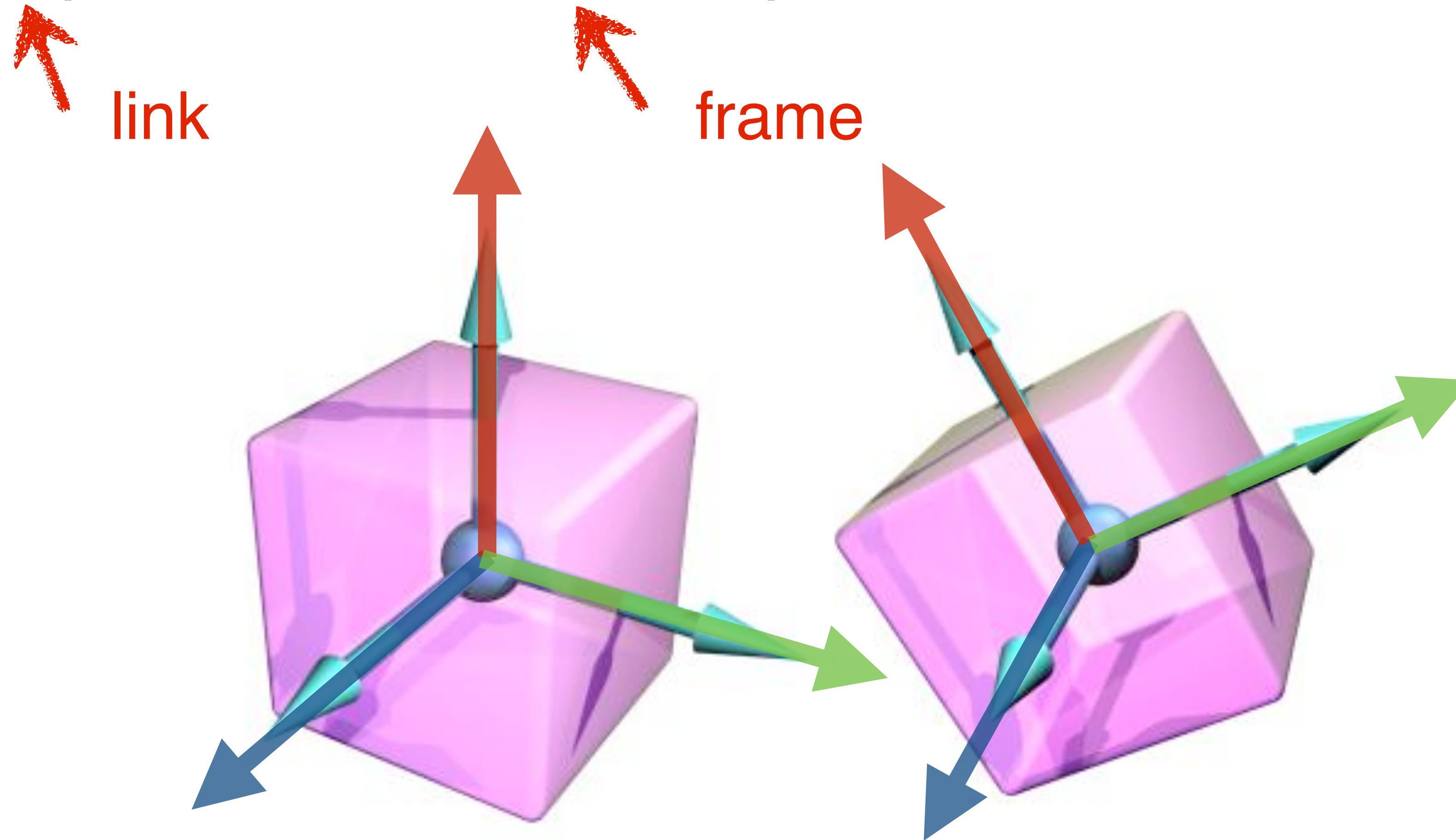
autorob.org





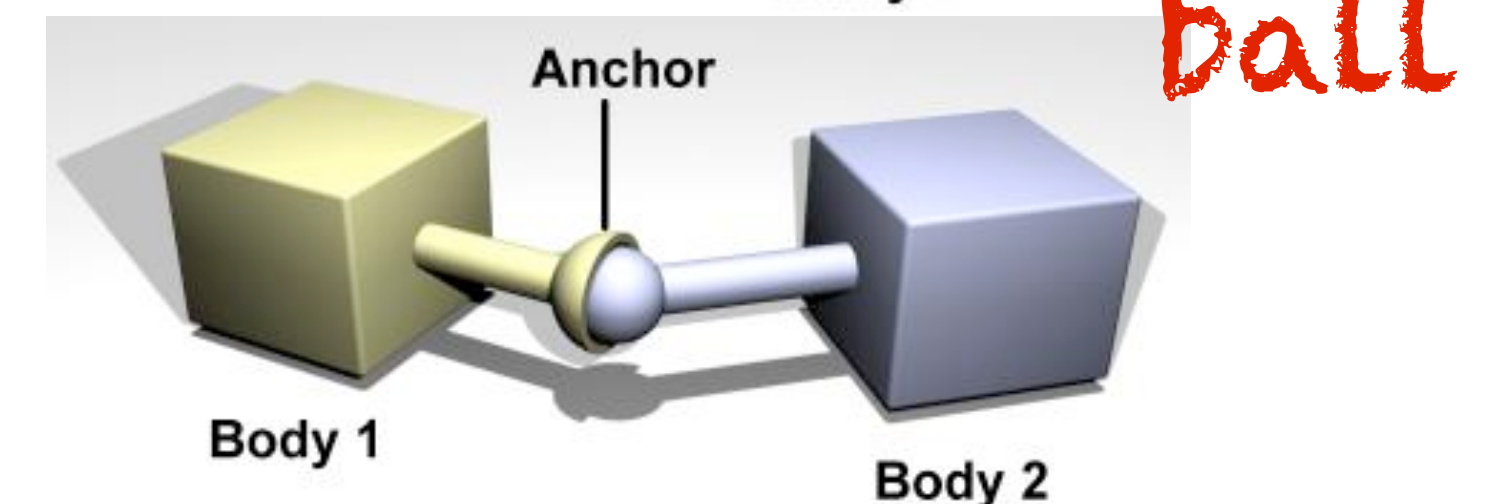
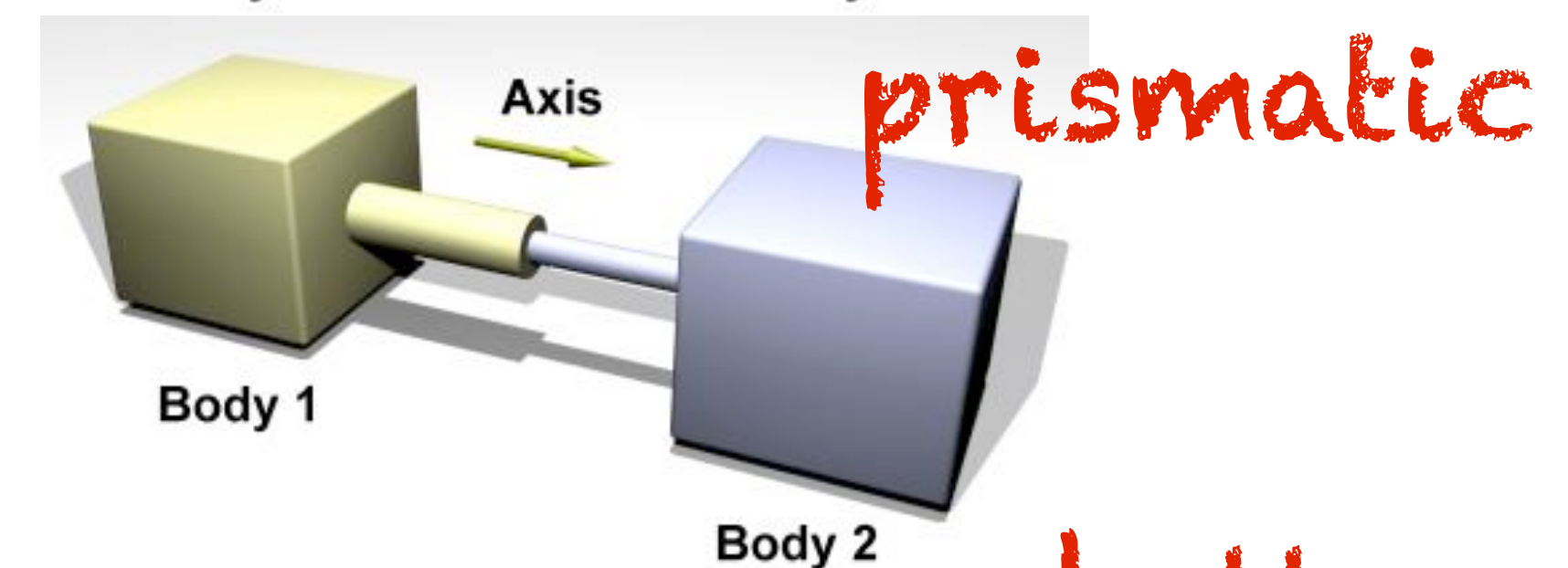
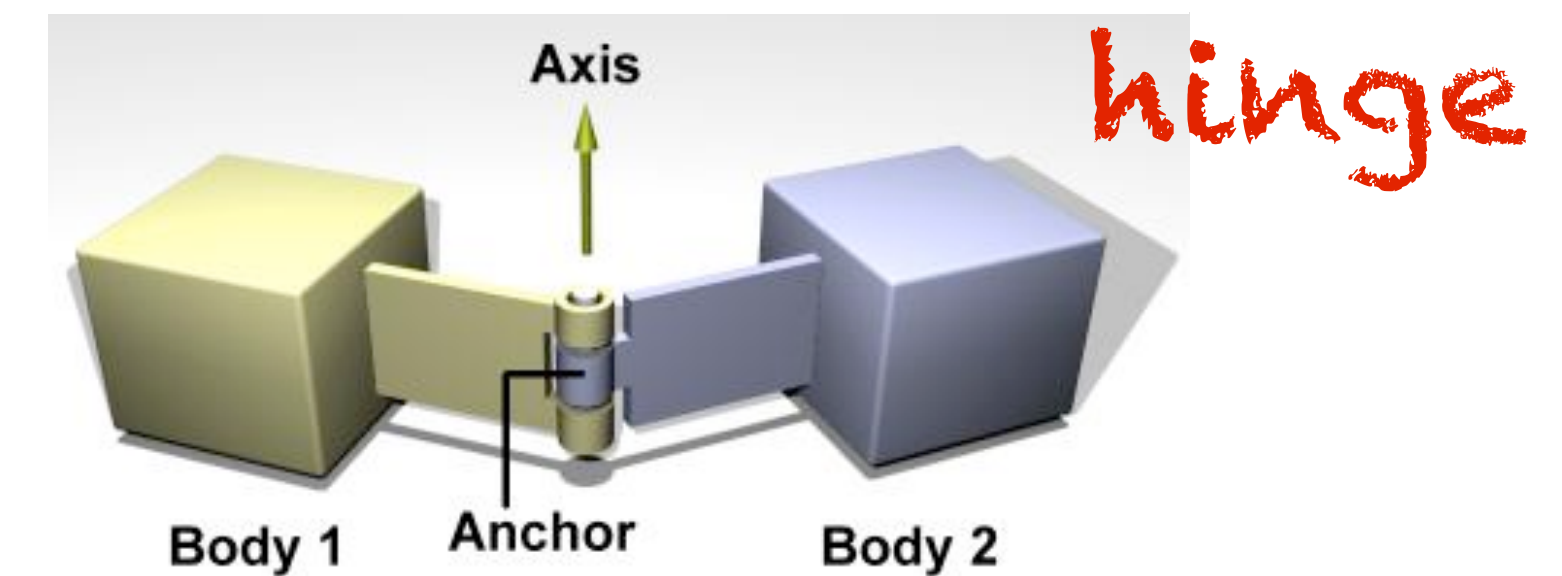
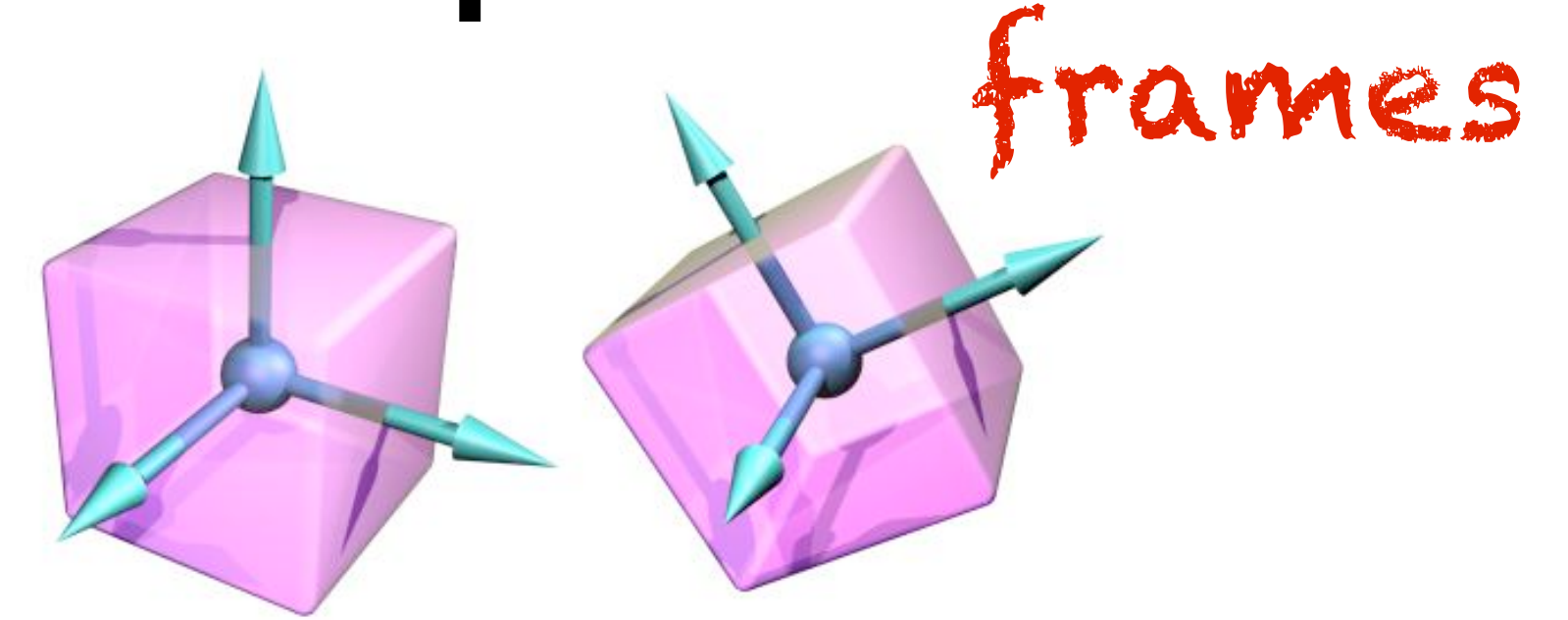
Reset: DOFs and Coordinate Spaces

- Each body has its own coordinate system



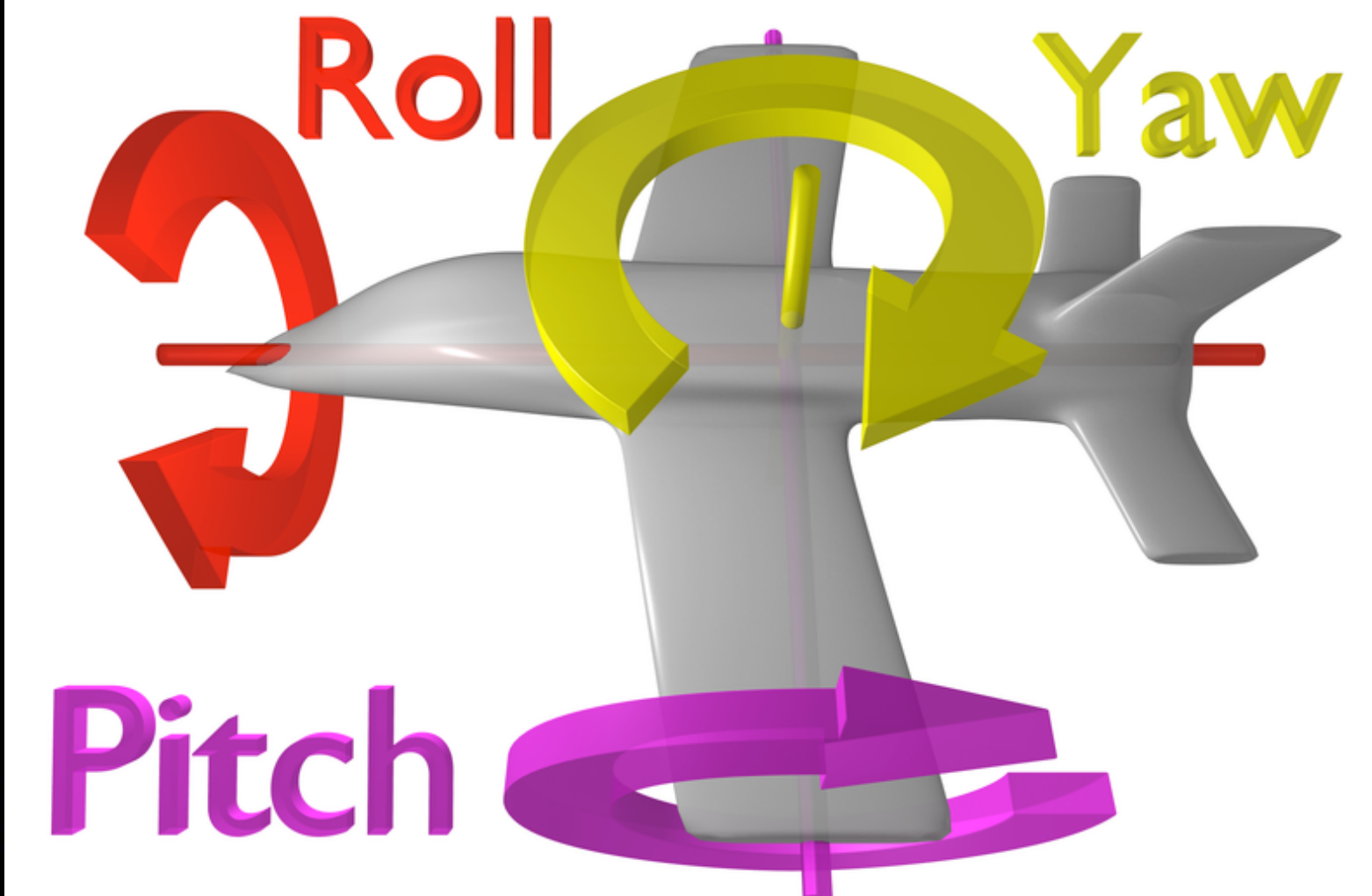
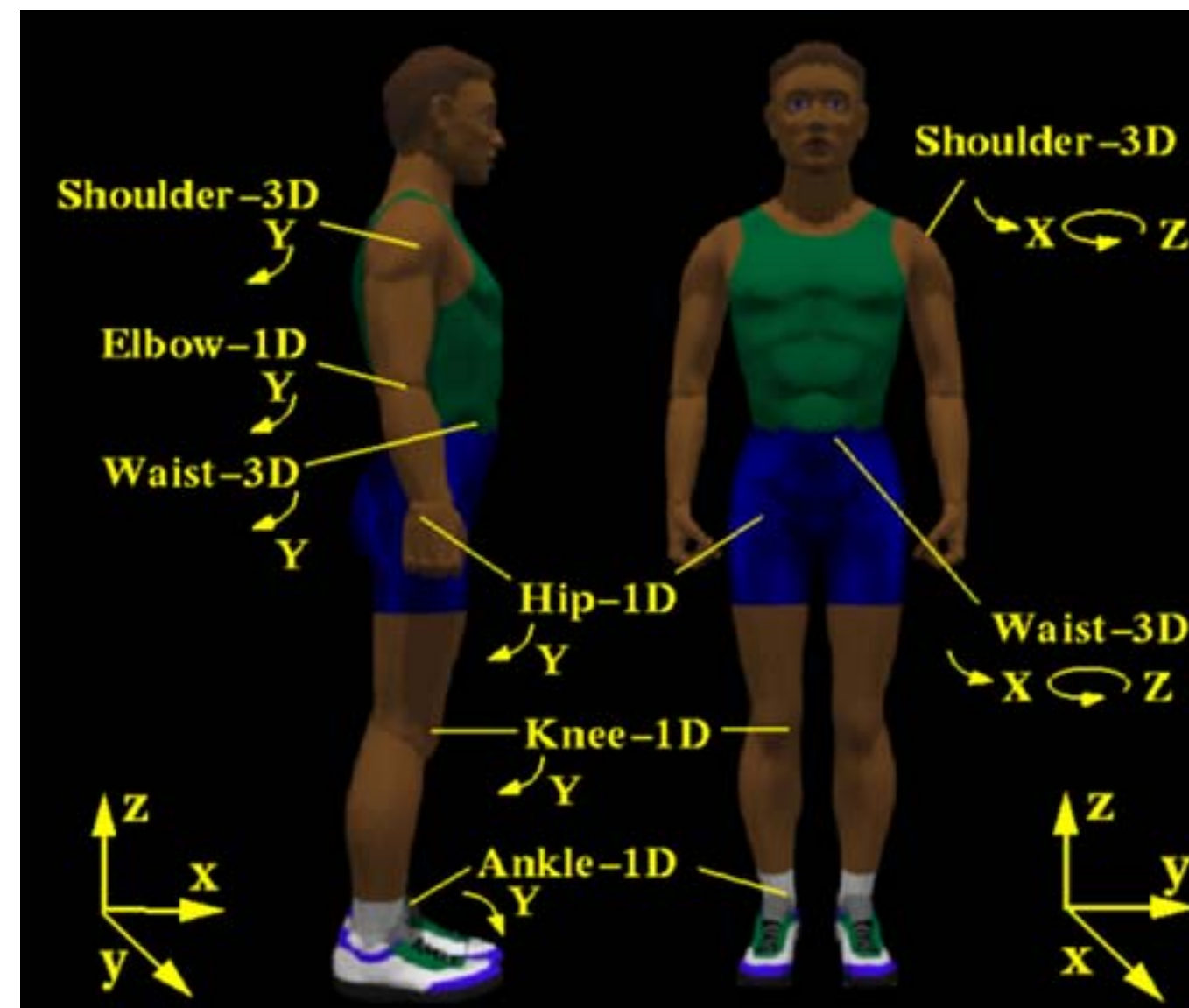
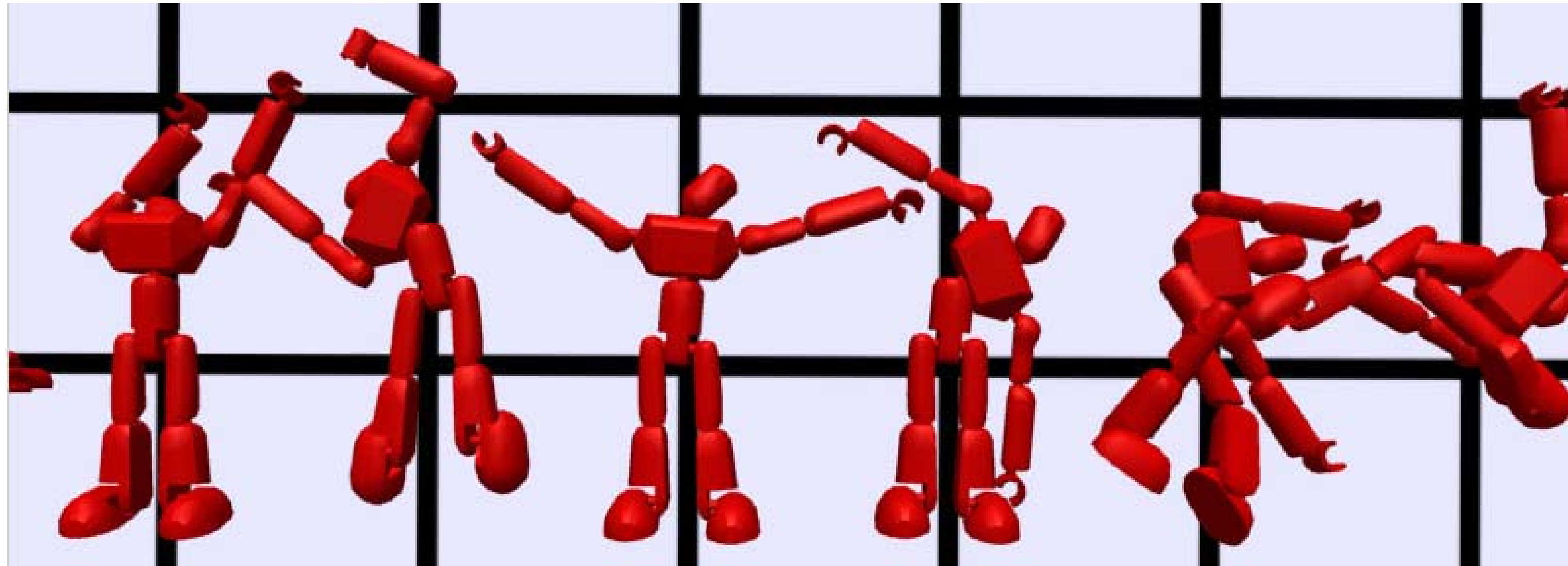
Reset: DOFs and Coordinate Spaces

- Each body has its own frame
 - Joints connect two links (rigid bodies)
 - e.g., hinge, prismatic, ball-socket
 - A motor exerts force on a DOF axis
- Linear algebra
- Matrix transformations used to relate coordinate systems of bodies and joints
 - Spatial geometry attached to each link, but does not affect the body's coordinate frame



Reset: Kinematics

- State comprised of degrees-of-freedom (DOFs)
- DOFs describe translation and rotation axes of system

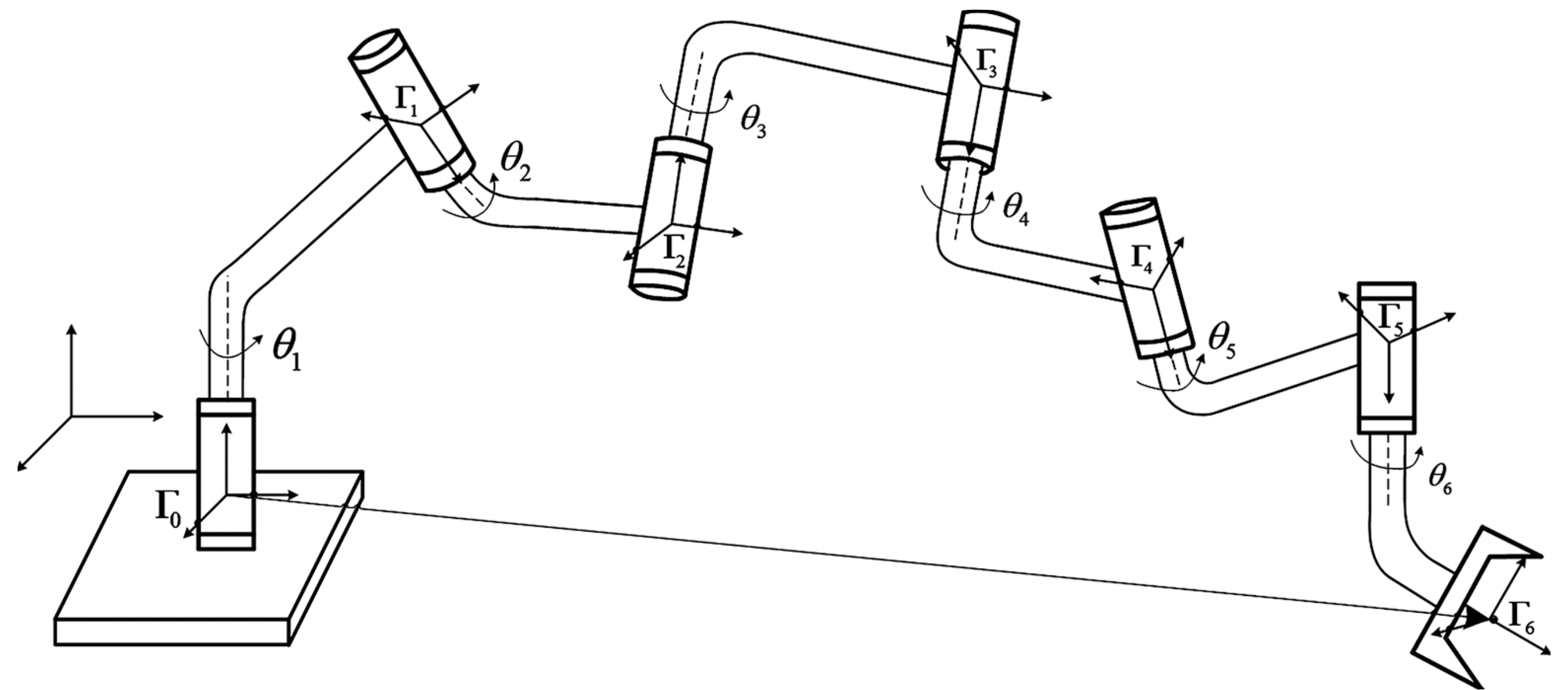


Robot Kinematics

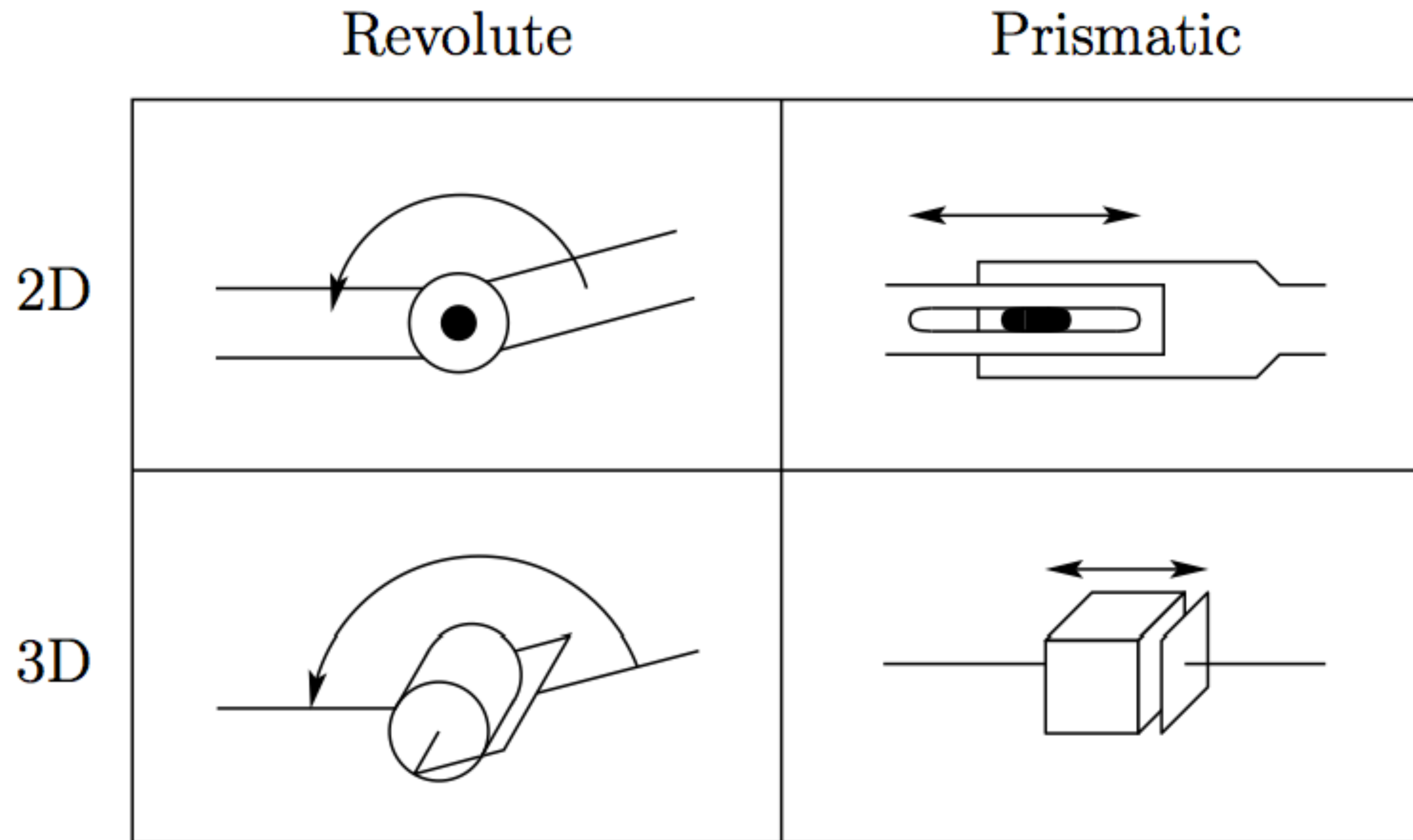
Goal: Given the structure of a robot arm, compute

- **Forward kinematics:** inferring the pose of the end-effector, given the state of each joint.
- **Inverse kinematics:** inferring the joint states necessary to reach a desired end-effector pose.

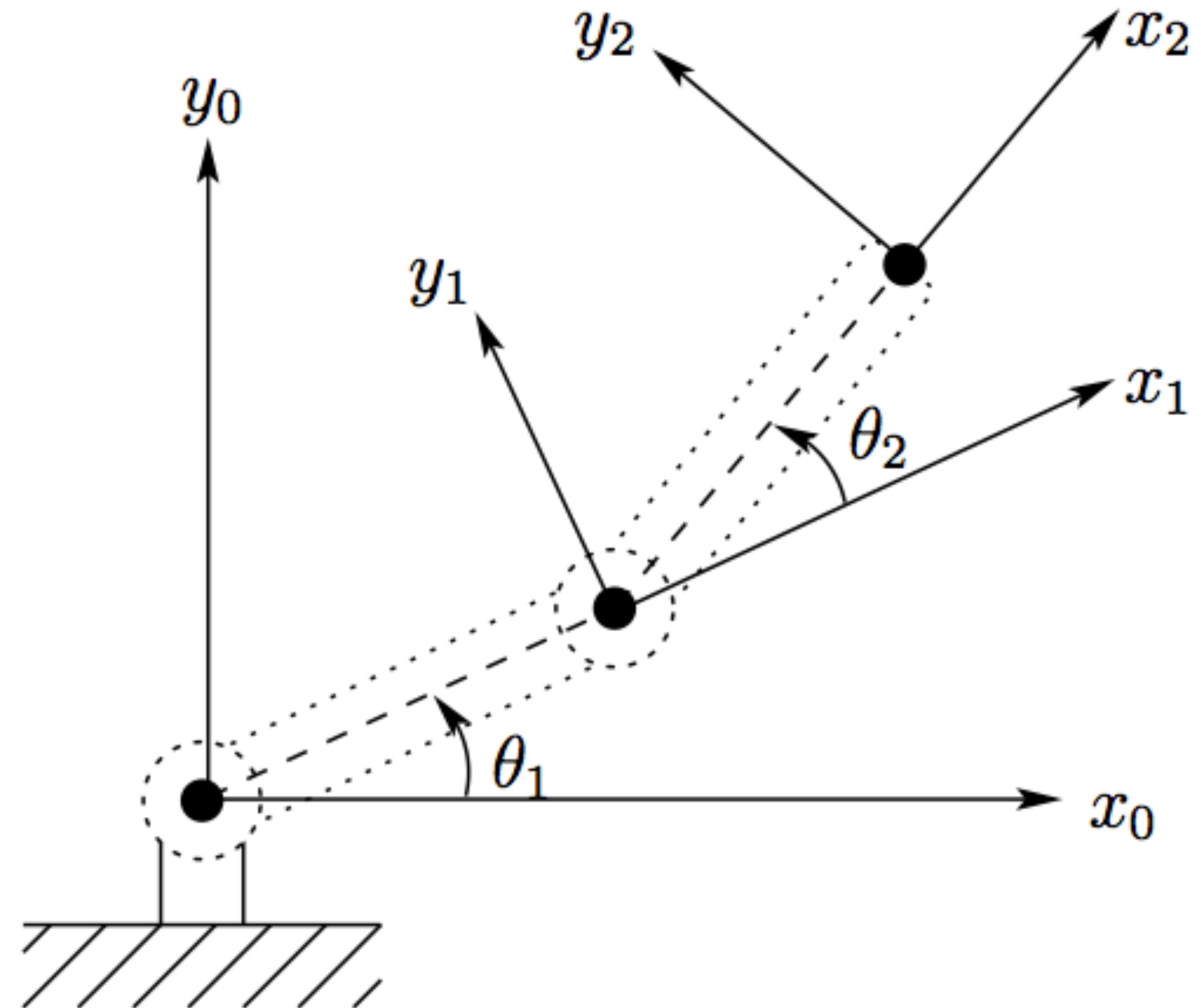
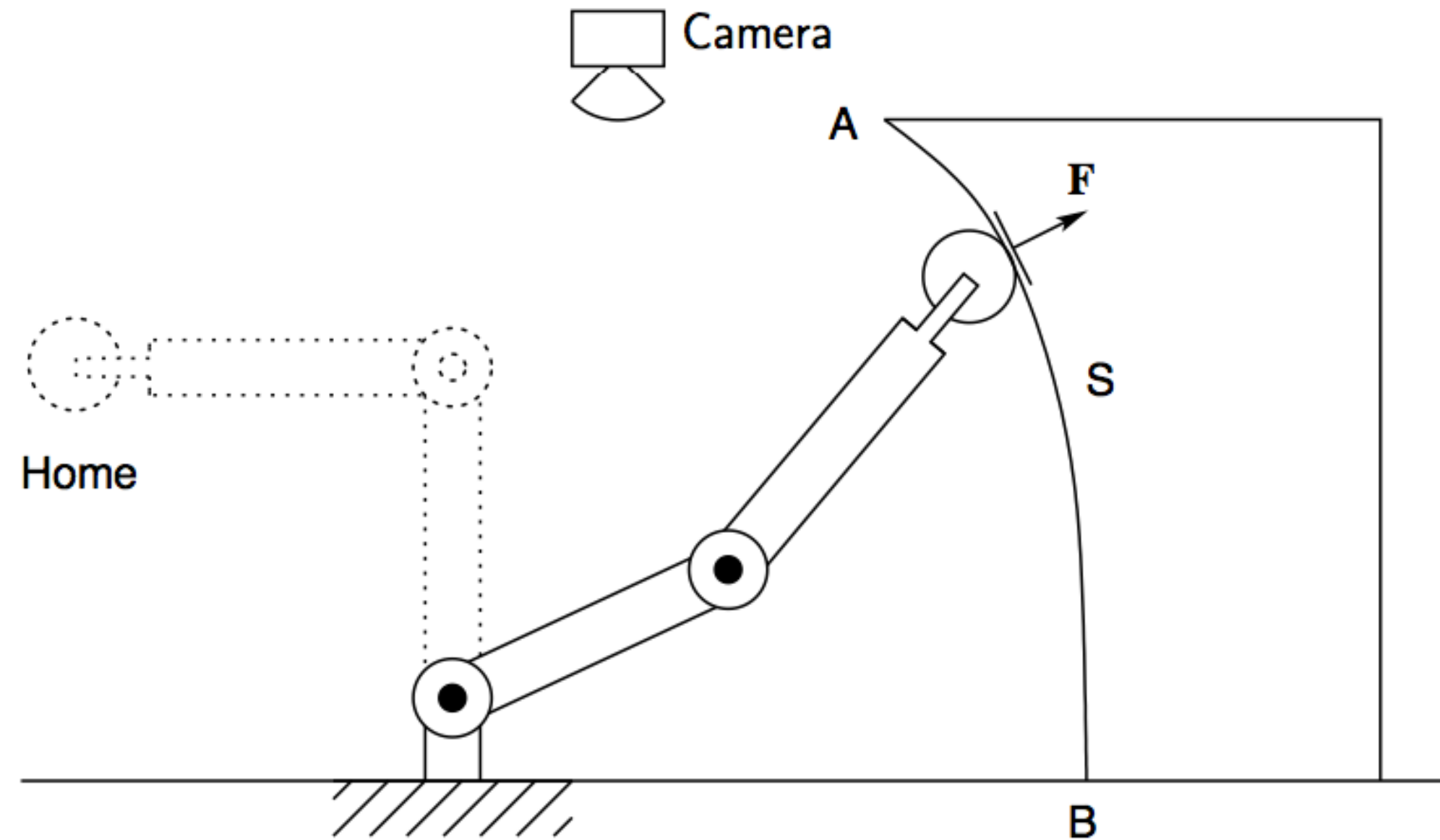
But, we need to start with a linear algebra refresher



DOF Visual Notation

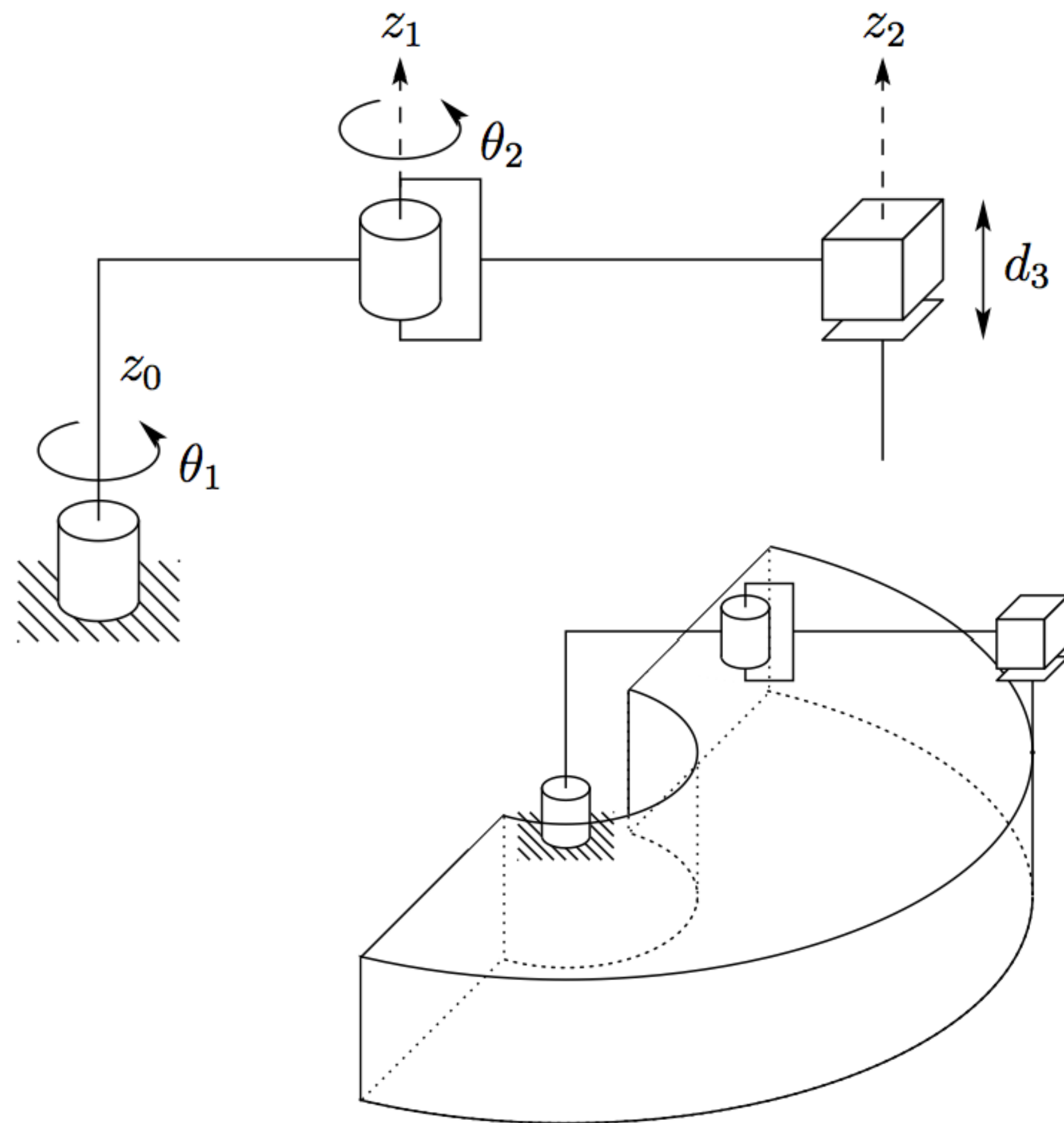


Planar 2-link Arm



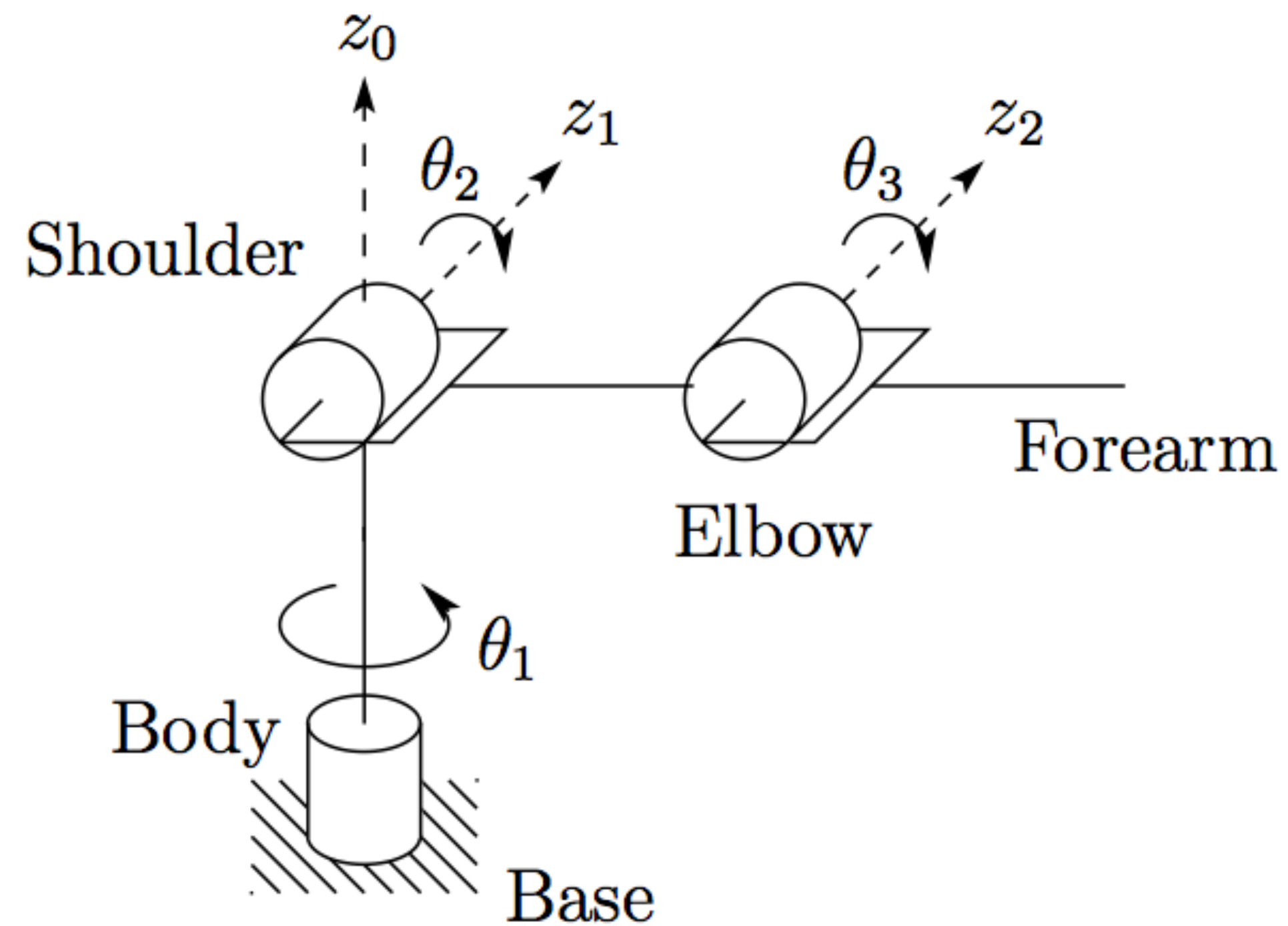
SCARA Arm

Selective Compliance Assembly Robot Arm



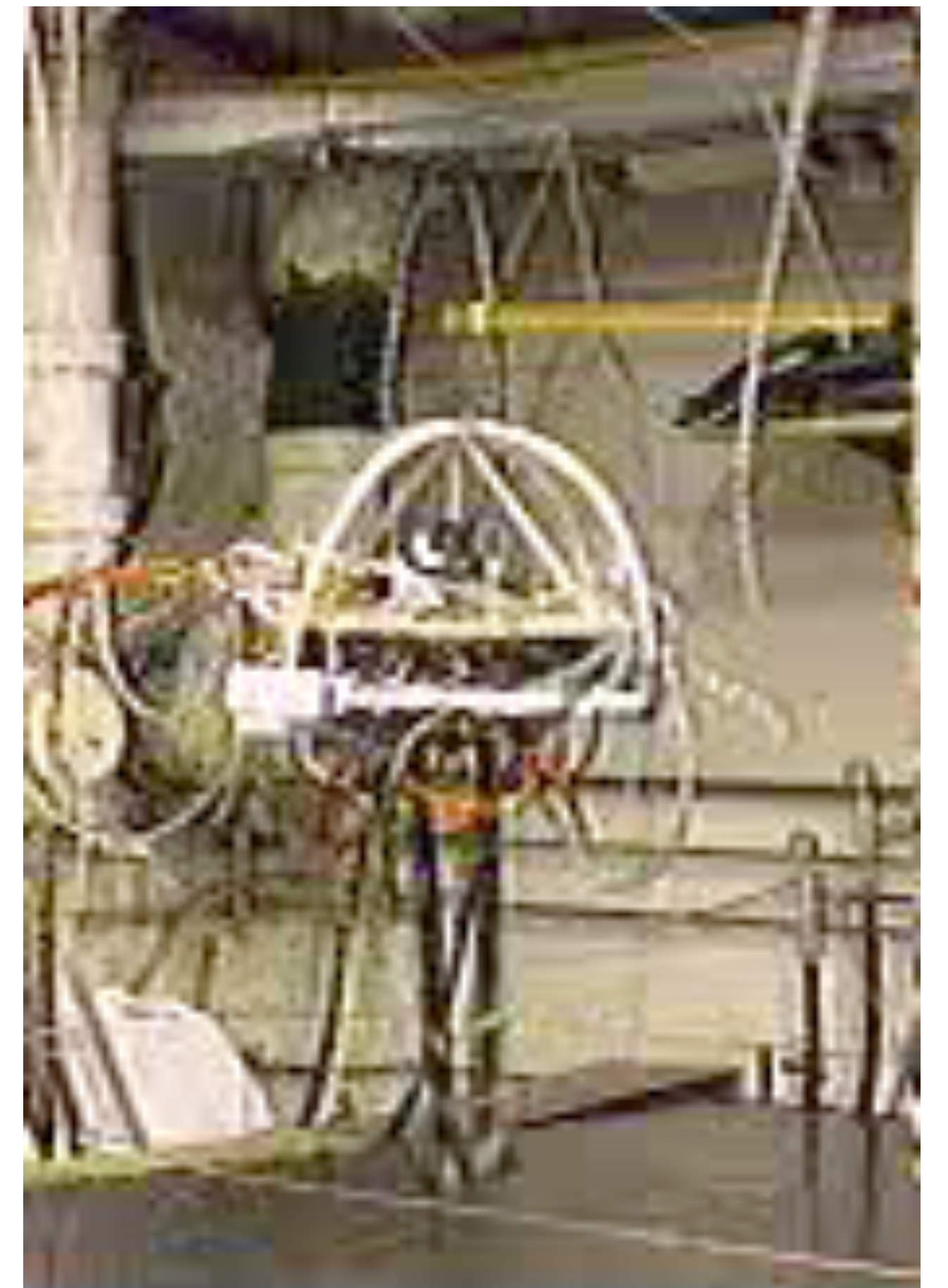
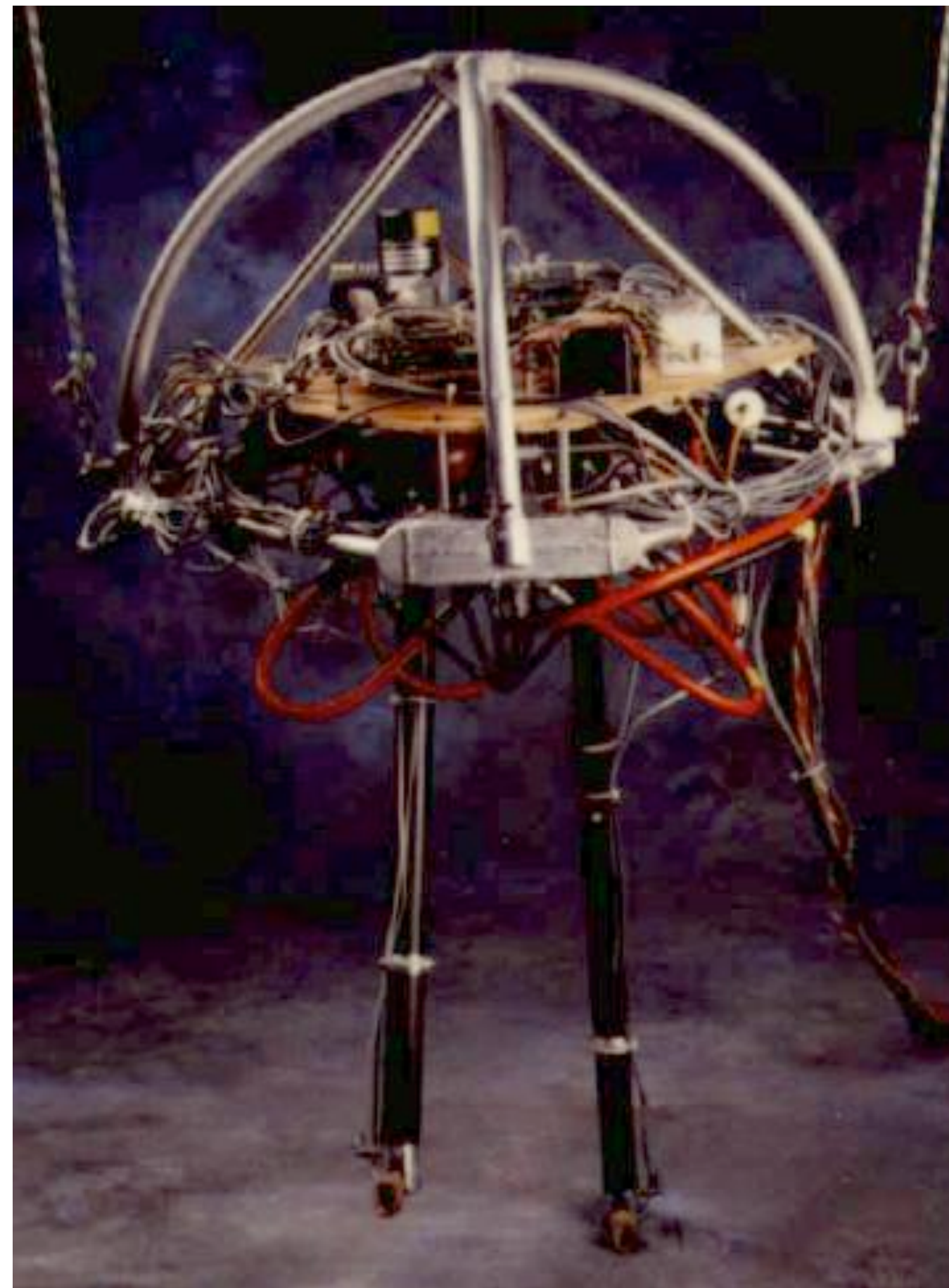
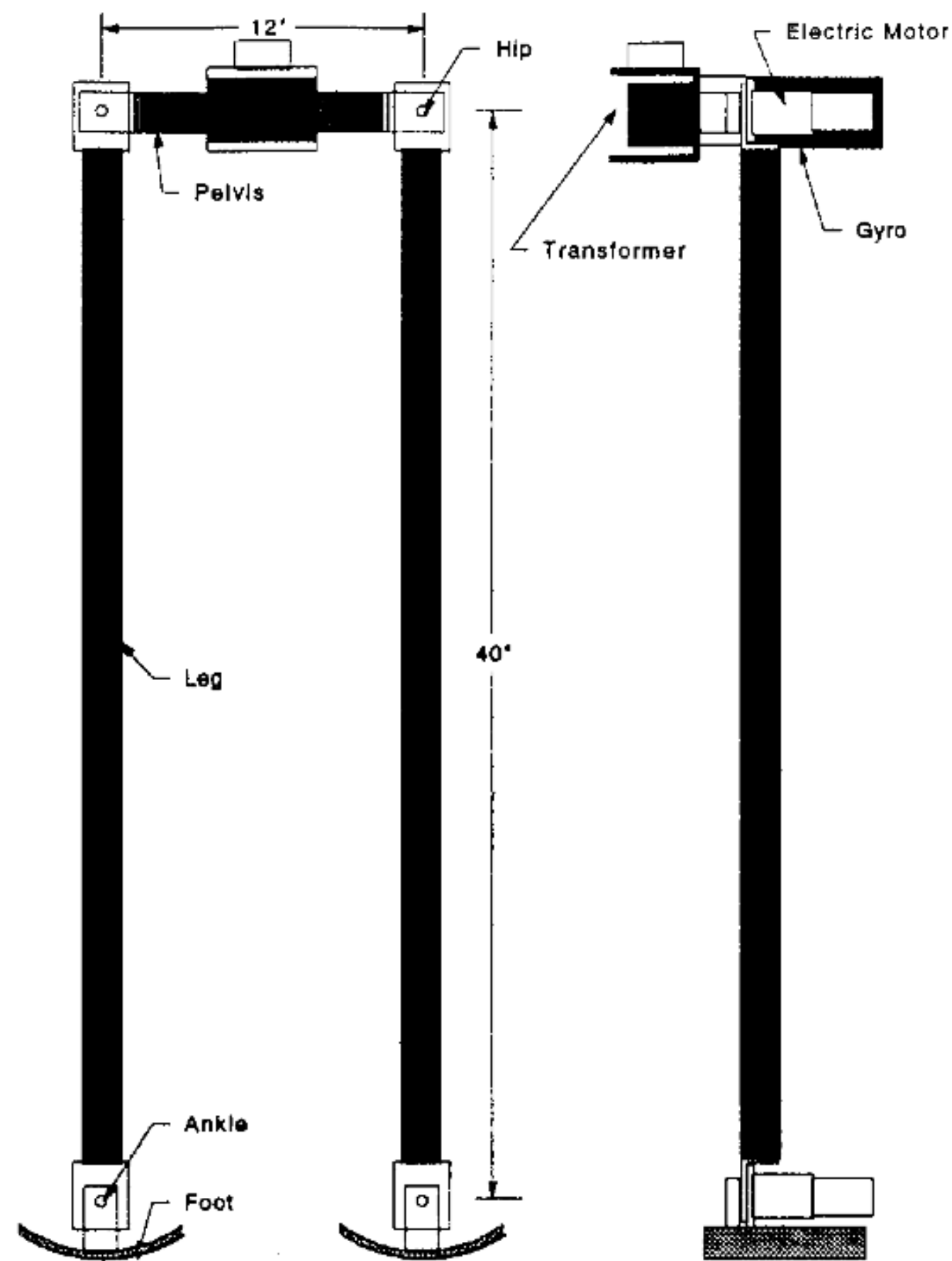
<https://youtu.be/7X5Nmk85kQo>

Motoman SK16



<https://youtu.be/Wj17z5iSzEQ>

Biped Hopper (MIT Leg Lab)

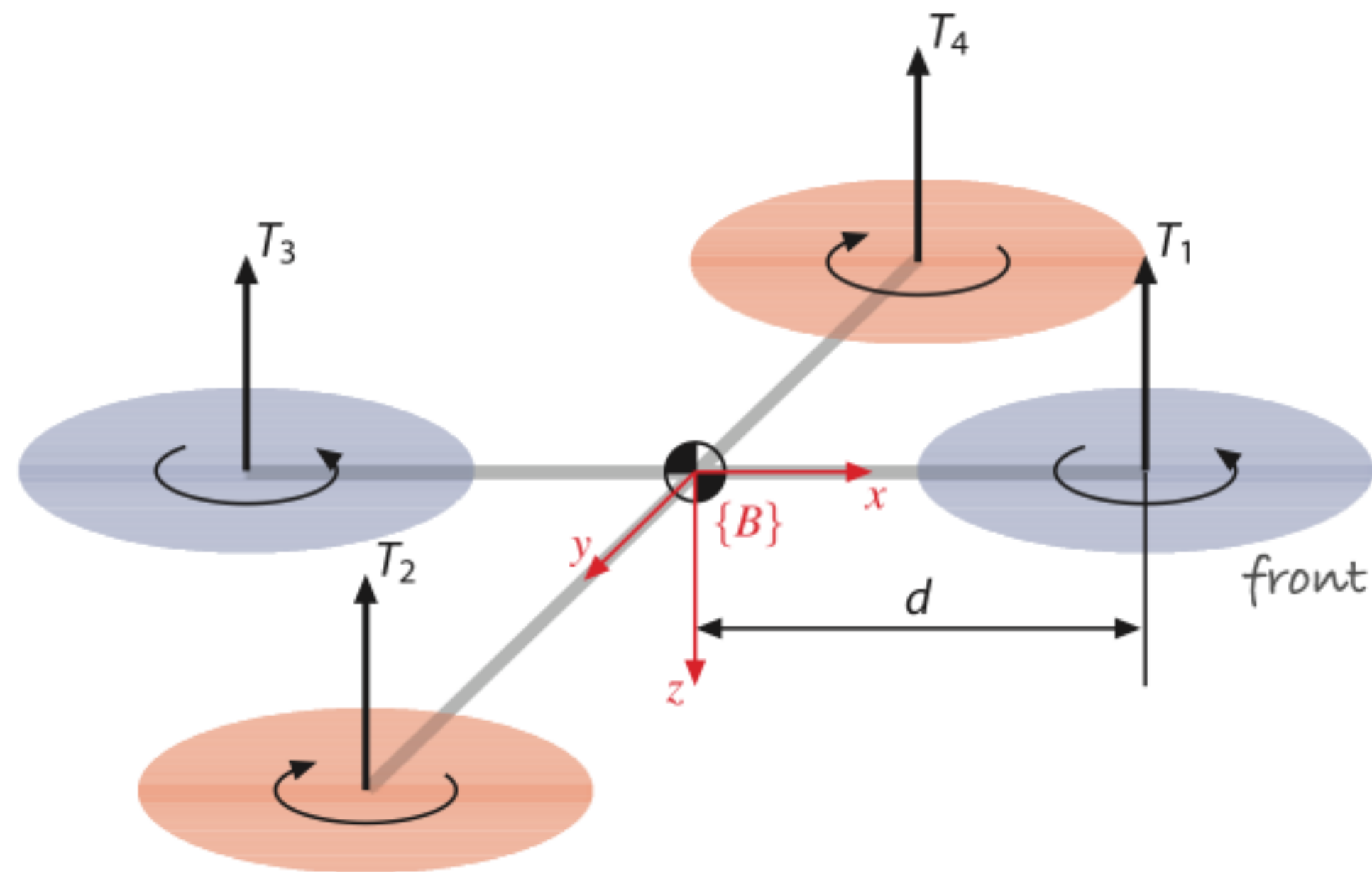


<http://www.ai.mit.edu/projects/leglab/robots/robots.html>

Big Dog (BDI)



Quad Rotor Helicopter



Safety is most important

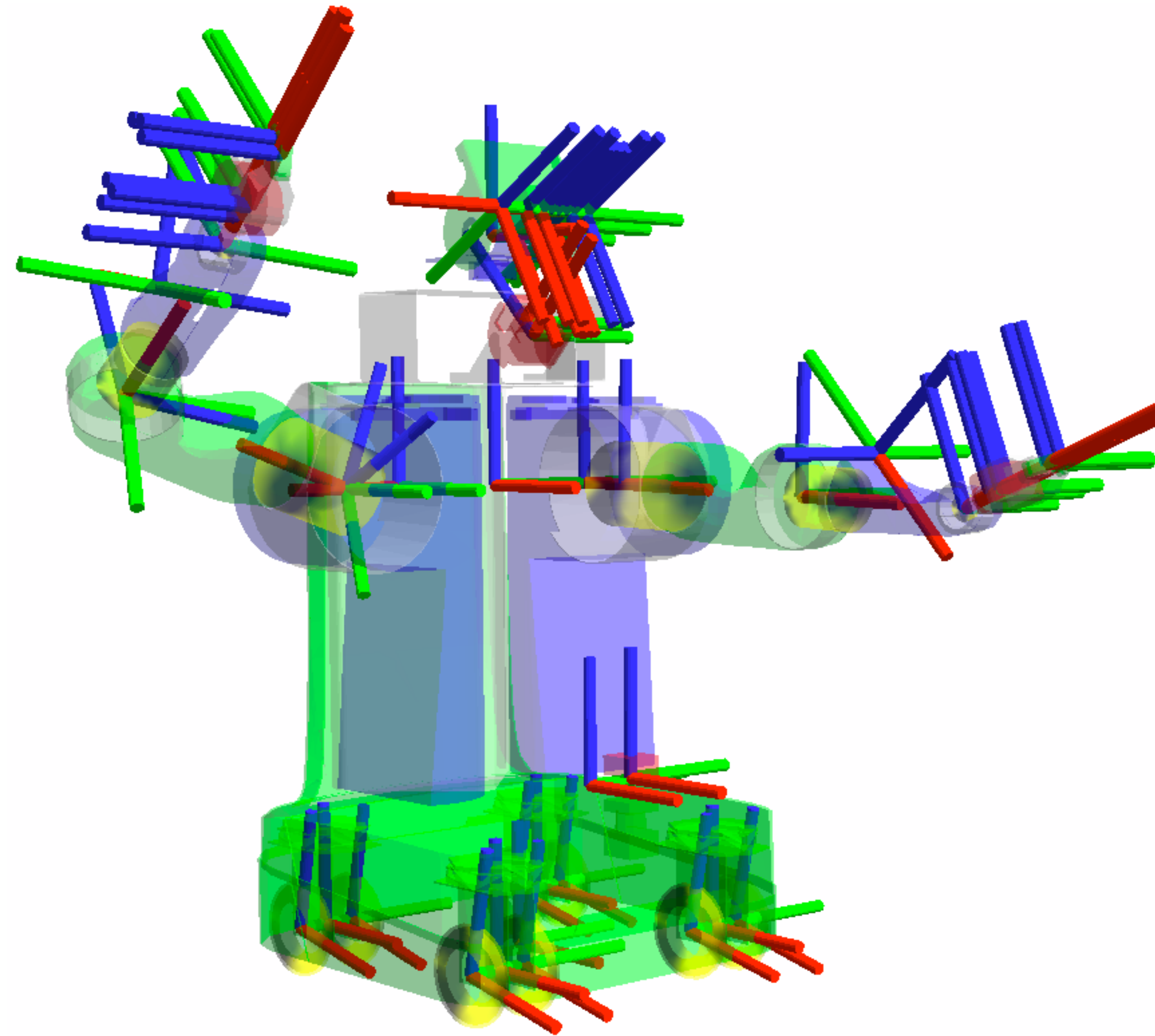
https://youtu.be/0mDiH_ajStQ



<https://www.youtube.com/watch?v=XxFZ-VStApo>



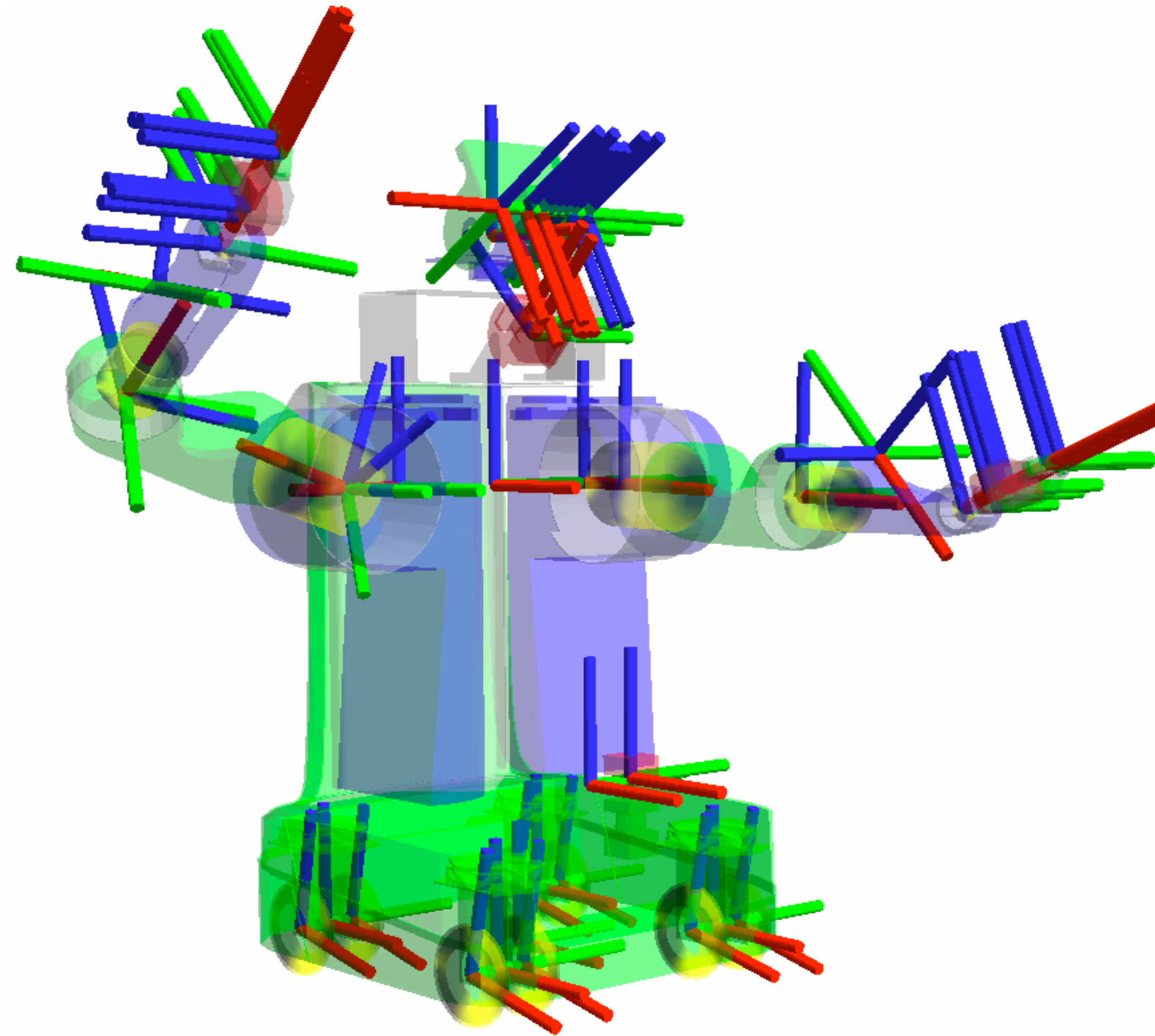
PR2



How to express kinematics as the state of an articulated system?



PR2



How to express kinematics as the state of an articulated system?

We need some math first.

Algebra

From Wikipedia, the free encyclopedia

Algebra (from [Arabic](#) "*al-jabr*" meaning "reunion of broken parts"^[1]) is one of the broad parts of [mathematics](#), together with [number theory](#), [geometry](#) and [analysis](#). In its most general form, algebra is the study of mathematical symbols and the rules for manipulating these symbols;^[2] it is a unifying thread of almost all of mathematics.^[3] As such, it includes everything from elementary equation solving to the study of abstractions such as [groups](#), [rings](#), and [fields](#). The more basic parts of algebra are

What does algebra provide
beyond arithmetic?

Algebra

From Wikipedia, the free encyclopedia

- Arithmetic applies to addition and multiplication of known numbers
- Algebra includes **abstractions as variables**
 - Unknown numbers or expressions that can take on many values
- An algebra supports addition and multiplication of variables and numbers.
 - For example, from: $x^2 = 5x - 6$
 - we get: $(x - 2)(x - 3) = 0$
 - and thus: $x = 2$ or $x = 3$.

Linear algebra

From Wikipedia, the free encyclopedia

Linear algebra is the branch of [mathematics](#) concerning [vector spaces](#) and [linear mappings](#) between such spaces. Such an investigation is initially motivated by a [system of linear equations](#) containing several unknowns. Such equations are naturally represented using the formalism of [matrices](#) and vectors.^[1]

What does linear algebra provide
beyond algebra?

Vector space

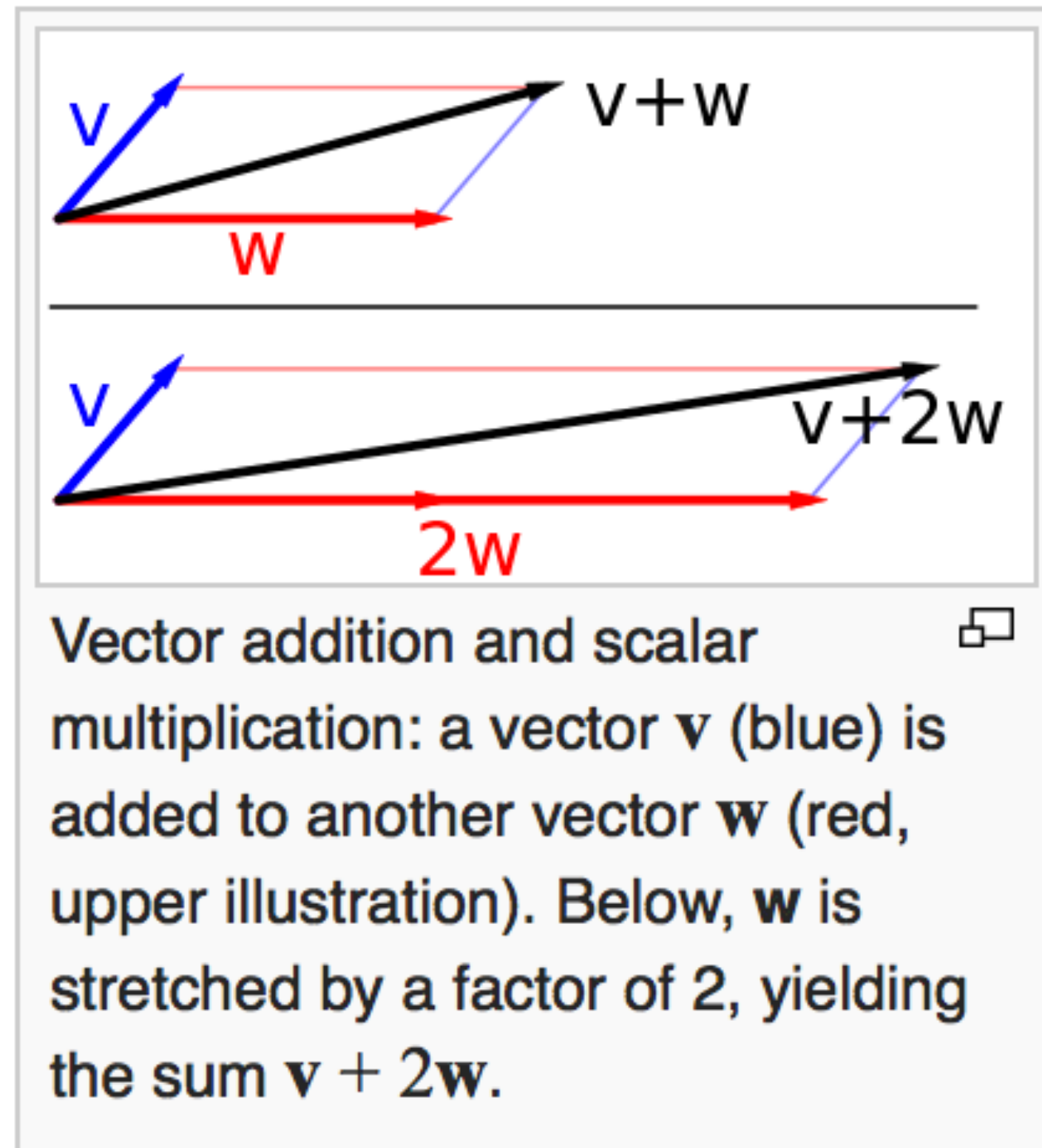
From Wikipedia, the free encyclopedia

This article is about linear (vector) spaces. For the structure in incidence geometry, see [Linear space \(geometry\)](#).

A **vector space** (also called a **linear space**) is a collection of objects called **vectors**, which may be [added](#) together and [multiplied](#) ("scaled") by numbers, called [scalars](#) in this context. Scalars are often taken to be [real numbers](#), but there are also vector spaces with scalar multiplication by [complex numbers](#), [rational numbers](#), or generally any [field](#). The operations of vector addition and scalar multiplication must satisfy certain requirements, called [axioms](#), listed [below](#).

- Describes spaces where vector operations are closed with respect to:

- a
- s



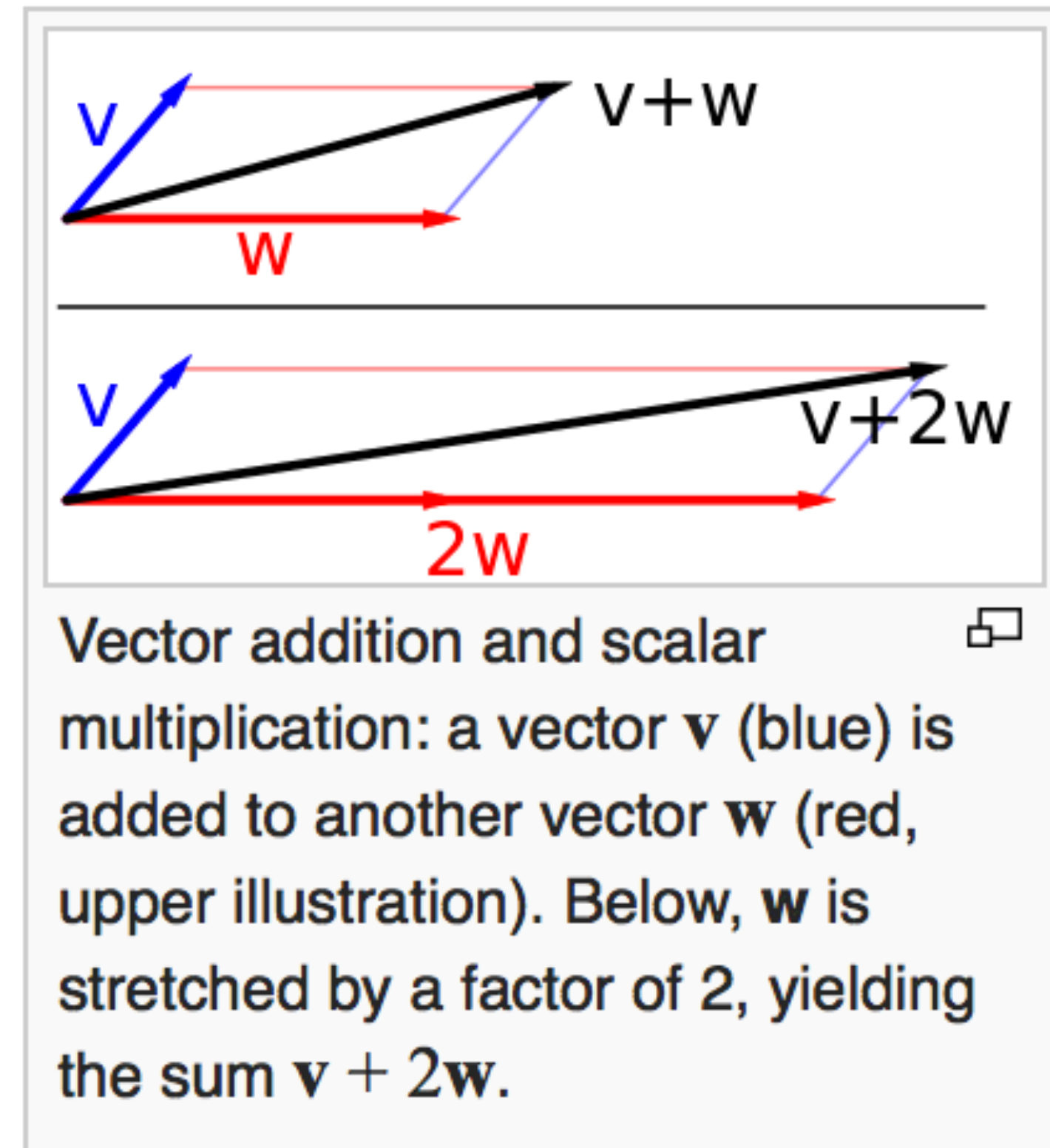
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- Describes spaces where vector operations are closed with respect to:
 - addition
 - scalar multiplication



	Arithmetic	Algebra	Linear Algebra
Abstraction		$x = 3$	$x = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$
Addition	$3 + 2 = 5$	$x + 2 = 5$	$x + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$
Scalar multiplication	$3 \times 2 = 6$	$2x = 6$	$2x = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$

Linear algebra

From Wikipedia, the free encyclopedia

- Many important complex systems are described by collections of linear equations.
- An algebra of scalars, vectors, and matrices helps us work with these systems, keeping track of the complexity.
- Manipulate groups of known and unknown parameters, just like manipulating numbers.
- Linear algebra is essential for representing frames of reference, rotation, translation, and general 3D homogeneous transforms.

Linear Algebra (Rough) Breakdown

- Geometry of Linear Algebra  primary focus for AutoRob
 - Vectors, matrices, basic operations, lines, planes, homogeneous coordinates, transformations
- Solving Linear Systems  needed for iterative IK
 - Gaussian Elimination, LU and Cholesky decomposition, over-determined systems, calculus and linear algebra, non-linear least squares, regression
- The Spectral Story
 - Eigensystems, singular value decomposition, principle component analysis, spectral clustering

Systems of Linear Equations

Linear algebra

From Wikipedia, the free encyclopedia

Linear algebra is the branch of [mathematics](#) concerning [vector spaces](#) and [linear mappings](#) between such spaces. Such an investigation is initially motivated by a [system of linear equations](#) containing several unknowns. Such equations are naturally represented using the formalism of [matrices](#) and vectors.^[1]

$$\begin{array}{rcl} 3x + 2y - z & = & 1 \\ 2x - 2y + 4z & = & -2 \\ -x + \frac{1}{2}y - z & = & 0 \end{array}$$

is solved by



Linear algebra

From Wikipedia, the free encyclopedia

Linear algebra is the branch of [mathematics](#) concerning [vector spaces](#) and [linear mappings](#) between such spaces. Such an investigation is initially motivated by a [system of linear equations](#) containing several unknowns. Such equations are naturally represented using the formalism of [matrices](#) and vectors.^[1]

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linear systems expressed
in general matrix form

$$A\mathbf{x} = \mathbf{b}$$

as

$$\begin{bmatrix} 3 & 2 & -1 \\ 2 & -2 & 4 \\ -1 & \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

M linear equations
each equation yields
a hyperplane in N-D


$$\begin{bmatrix} 3 & 2 & -1 \\ 2 & -2 & 4 \\ -1 & \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$A \quad \mathbf{x} = \mathbf{b}$$



vector of N unknowns to be found

M linear equations

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a hyperplane in N-D


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$$A \mathbf{x} = \mathbf{b}$$



vector of N unknowns to be found

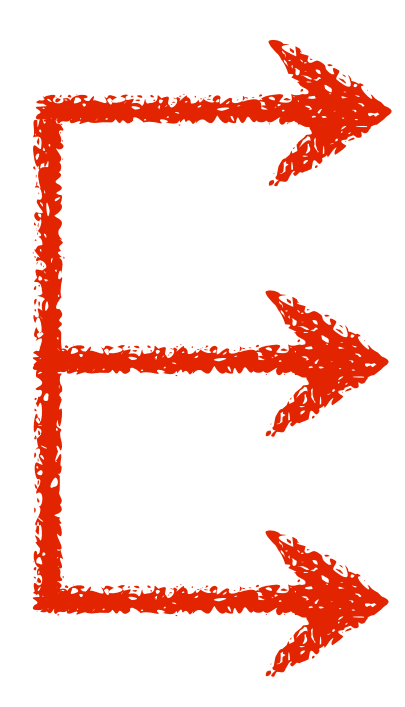
If #unknowns > #equations,

If #unknowns < #equations,

If #unknowns = #equations,

M linear equations

each equation yields
a hyperplane in N-D


$$\begin{bmatrix} 3 & 2 & -1 \\ 2 & -2 & 4 \\ -1 & \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$A \mathbf{x} = \mathbf{b}$$



vector of N unknowns to be found

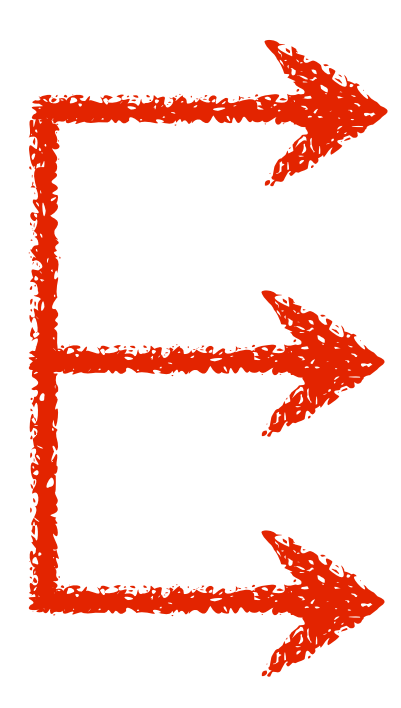
If #unknowns > #equations, underdetermined system, usually with infinite solutions

If #unknowns < #equations,

If #unknowns = #equations,

M linear equations

each equation yields
a hyperplane in N-D


$$\begin{bmatrix} 3 & 2 & -1 \\ 2 & -2 & 4 \\ -1 & \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$A \mathbf{x} = \mathbf{b}$$



vector of N unknowns to be found

If #unknowns > #equations, underdetermined system, usually with infinite solutions

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If #unknowns = #equations,

M linear equations

each equation yields
a hyperplane in N-D


$$\begin{bmatrix} 3 & 2 & -1 \\ 2 & -2 & 4 \\ -1 & \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$A \mathbf{x} = \mathbf{b}$$



vector of N unknowns to be found

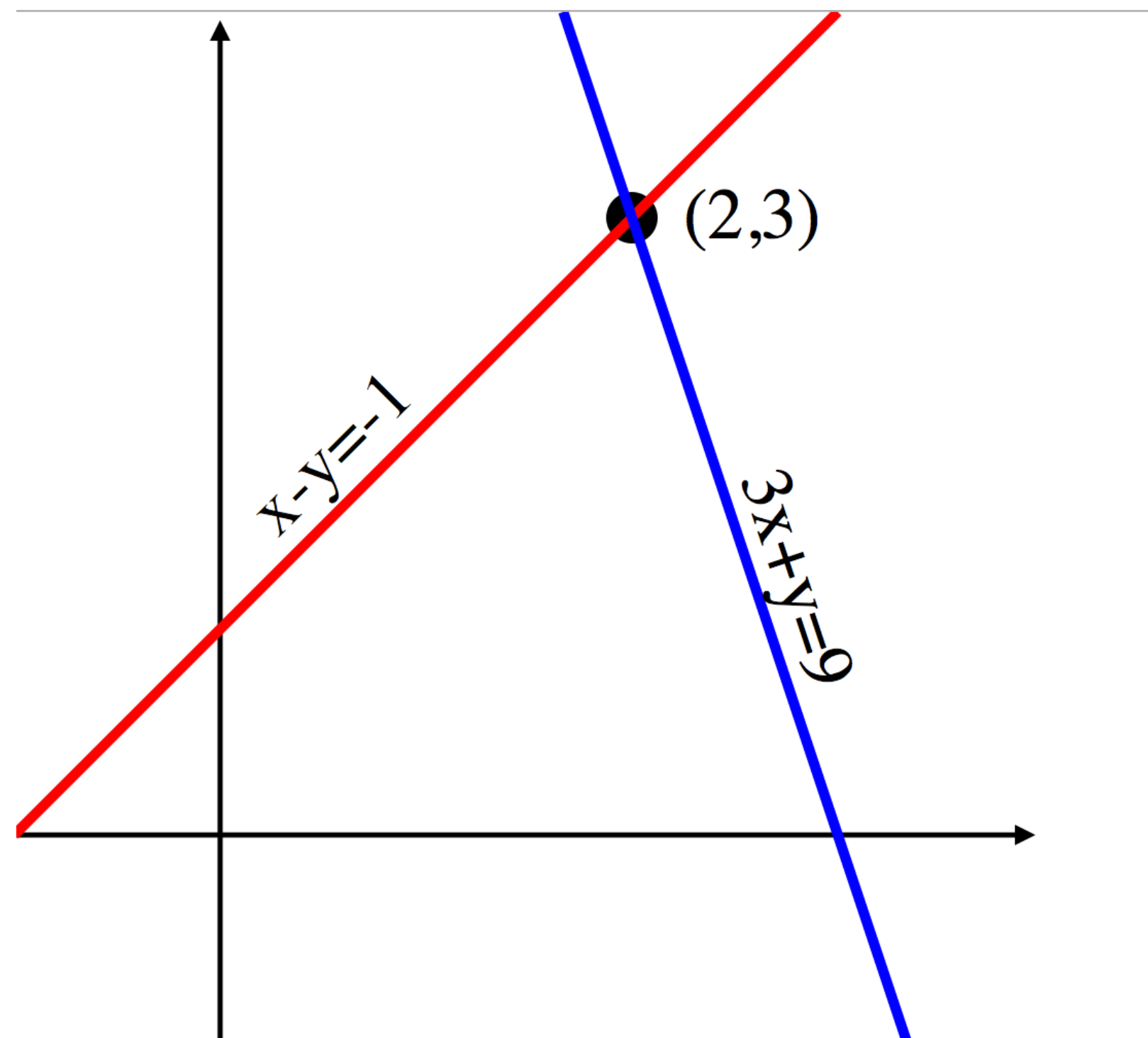
If #unknowns > #equations, underdetermined system, usually with infinite solutions

If #unknowns < #equations, overdetermined system, usually with no solutions

If #unknowns = #equations, usually has a unique solution

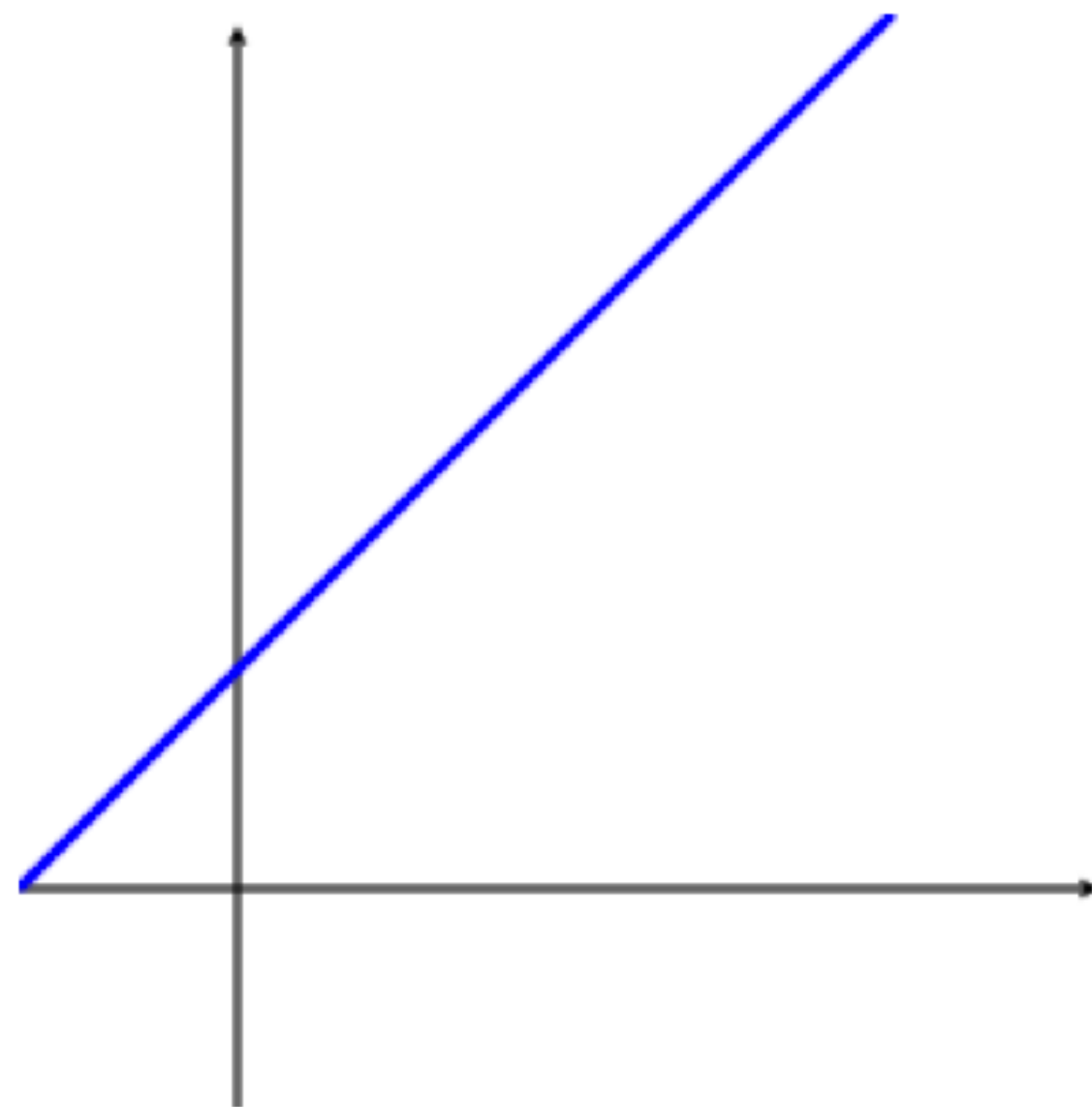
2D Example

only single point
satisfies both lines



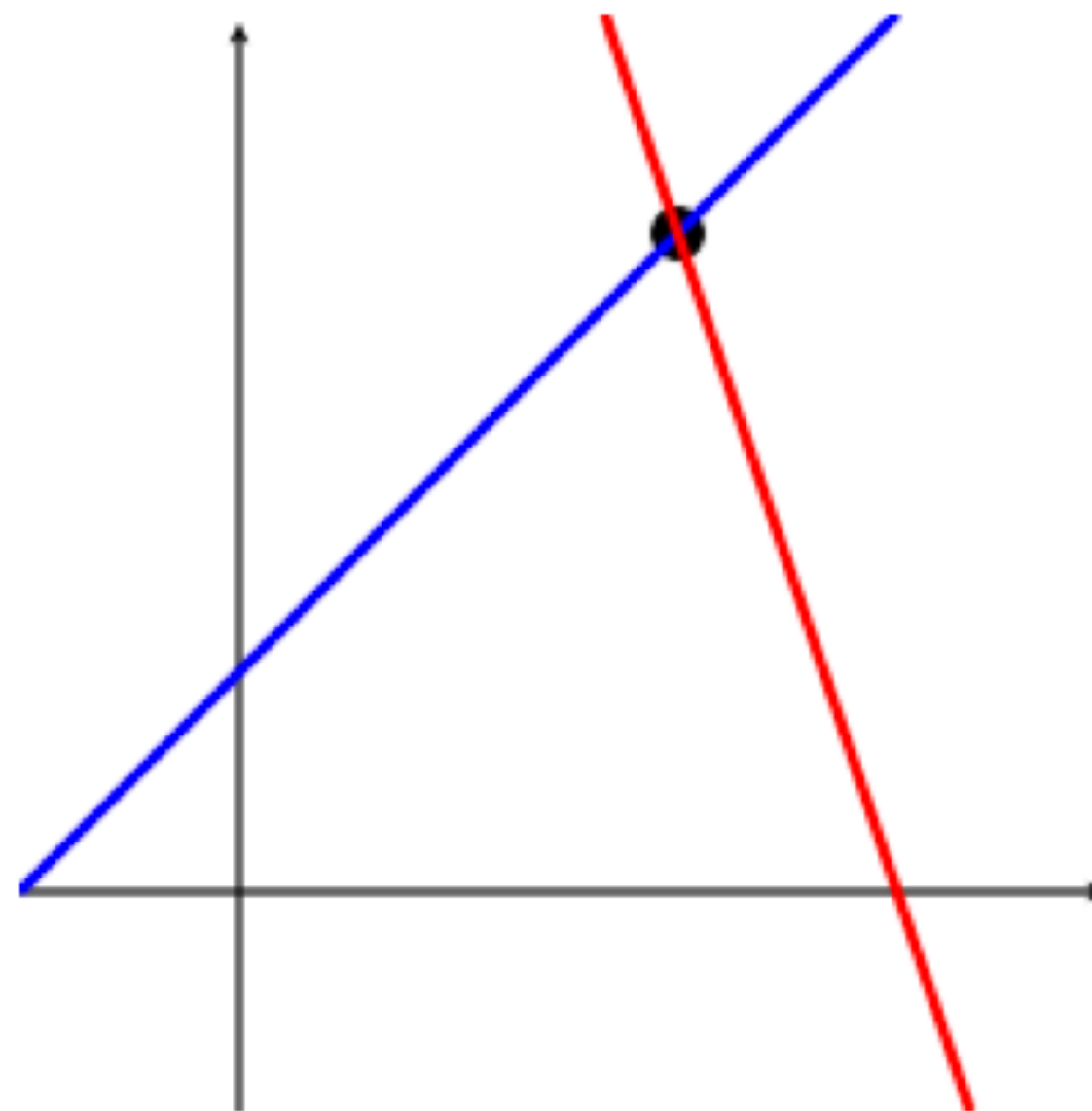
2D Example

any point on the
line satisfies



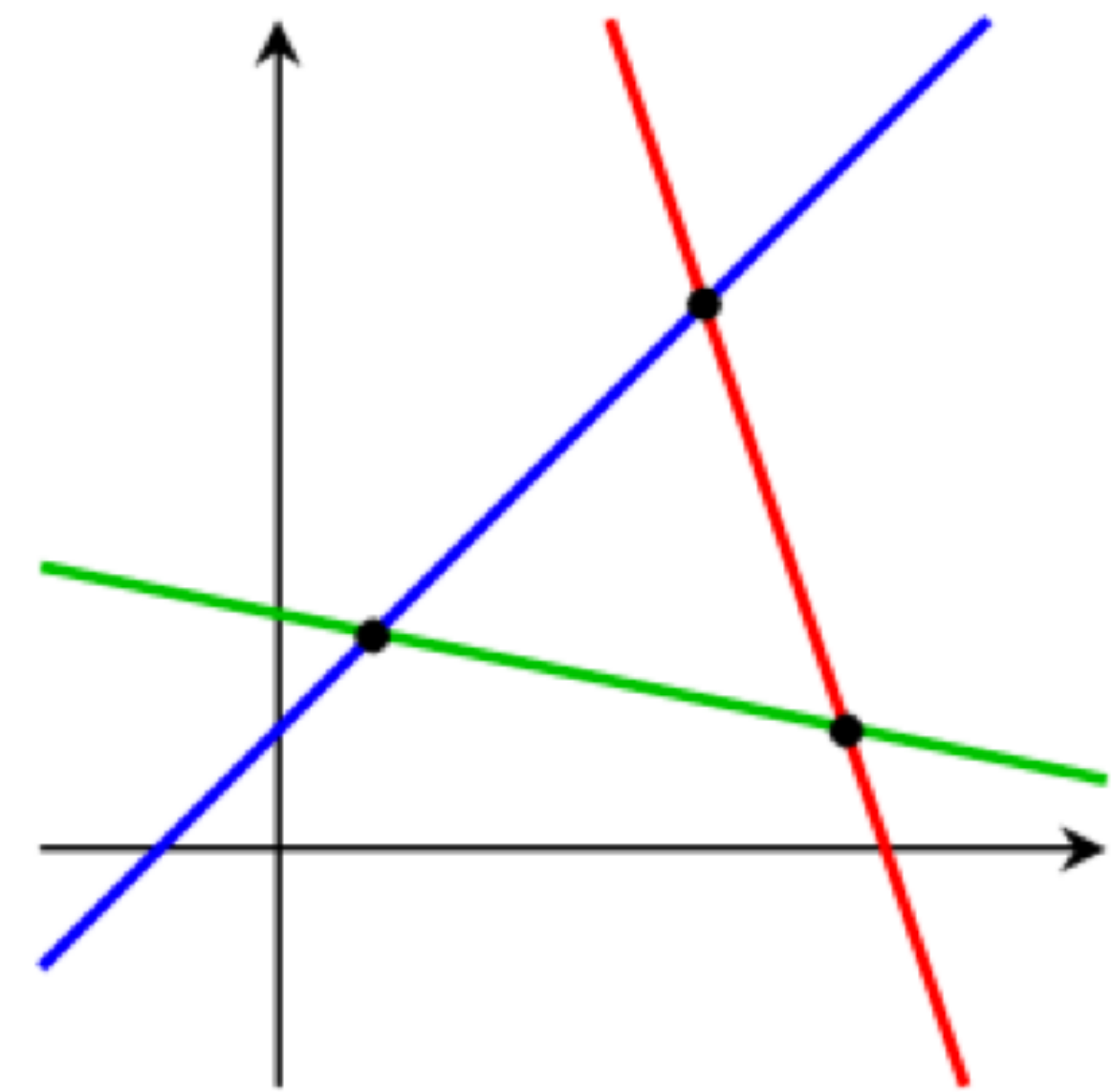
One equation

only single point
satisfies both lines



Two equations

no point satisfies
all three lines

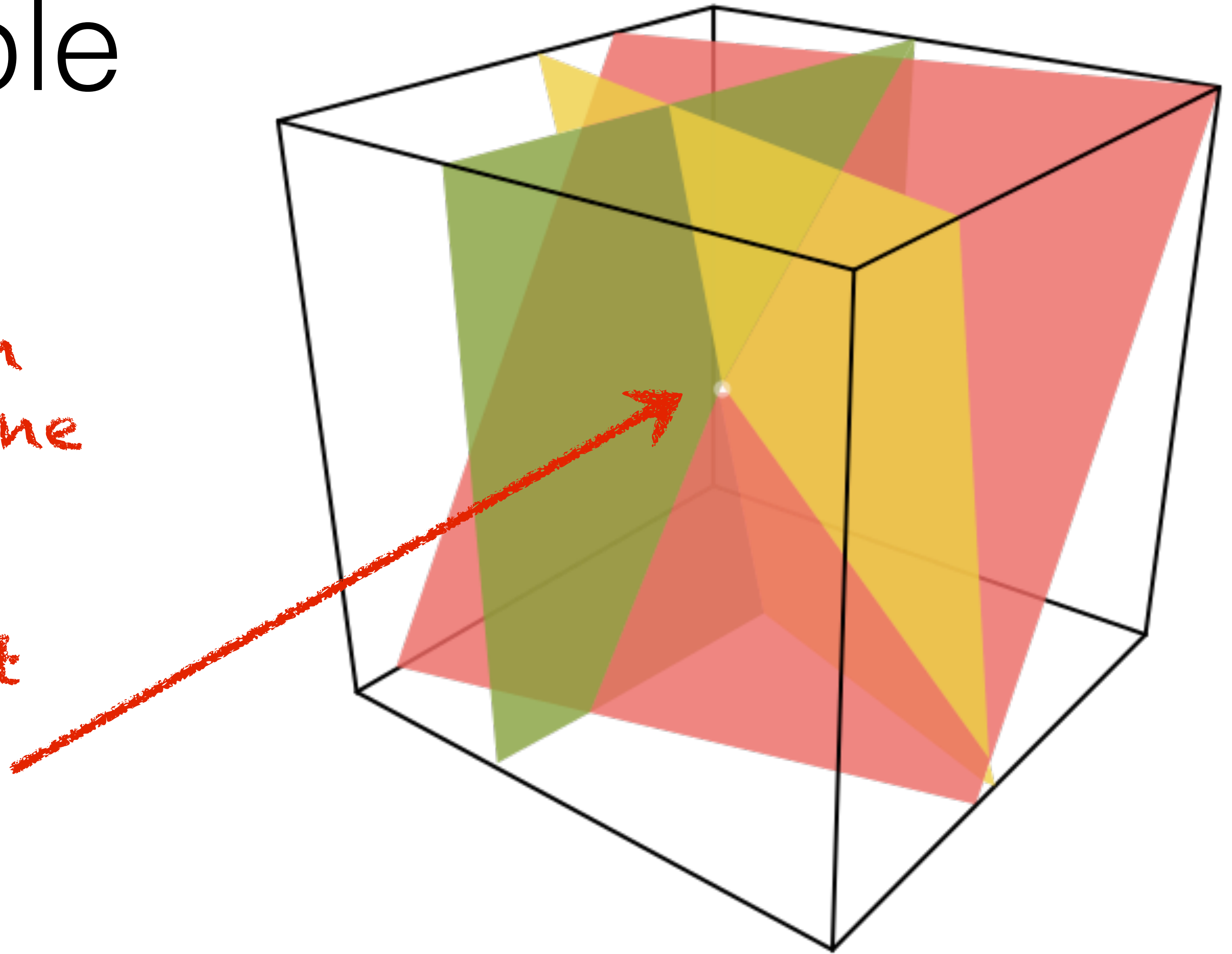


Three equations

3D Example

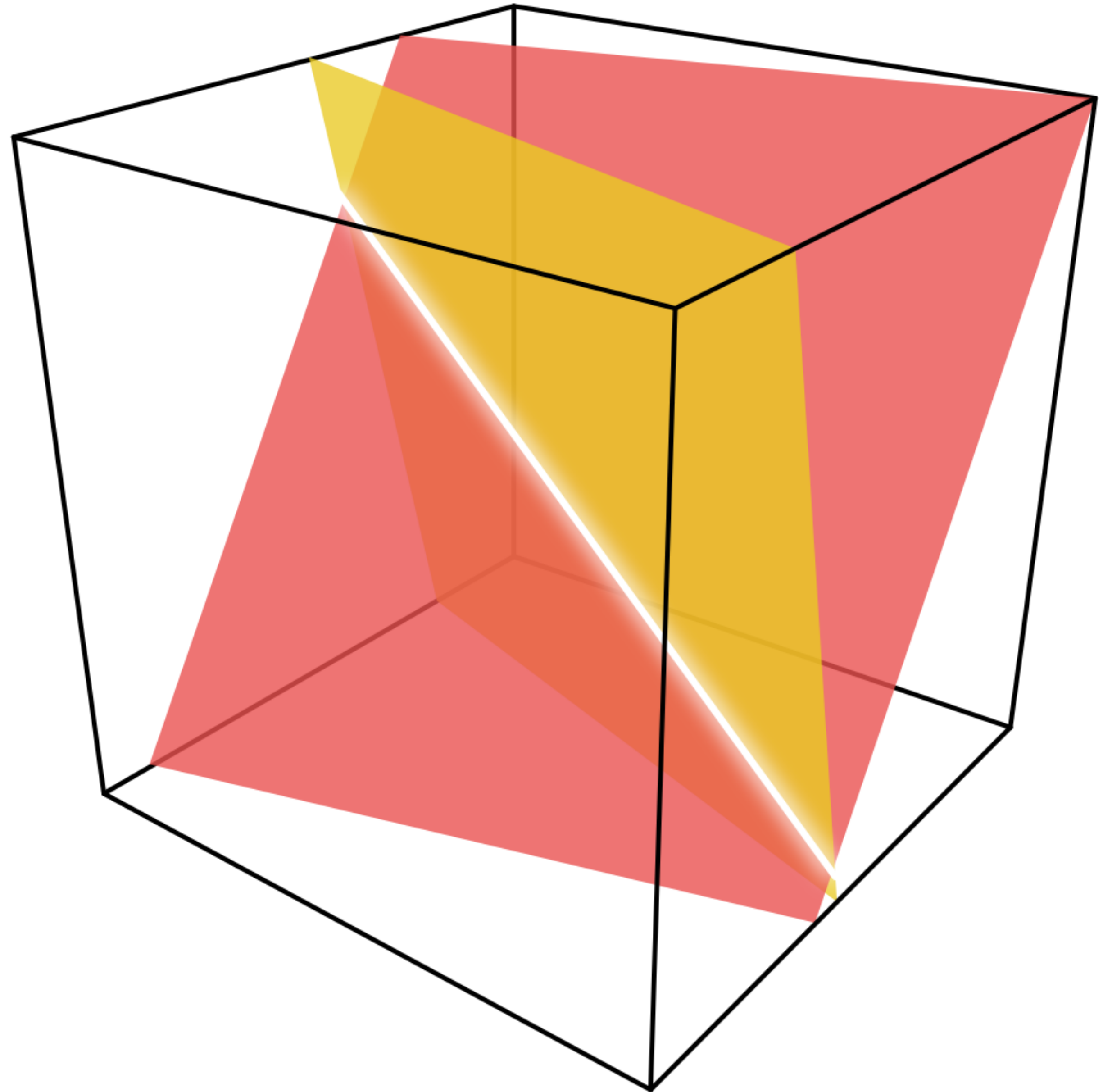
Each equation
yields a 2D plane
in 3D space

A single point
satisfies all
equations



3D Example

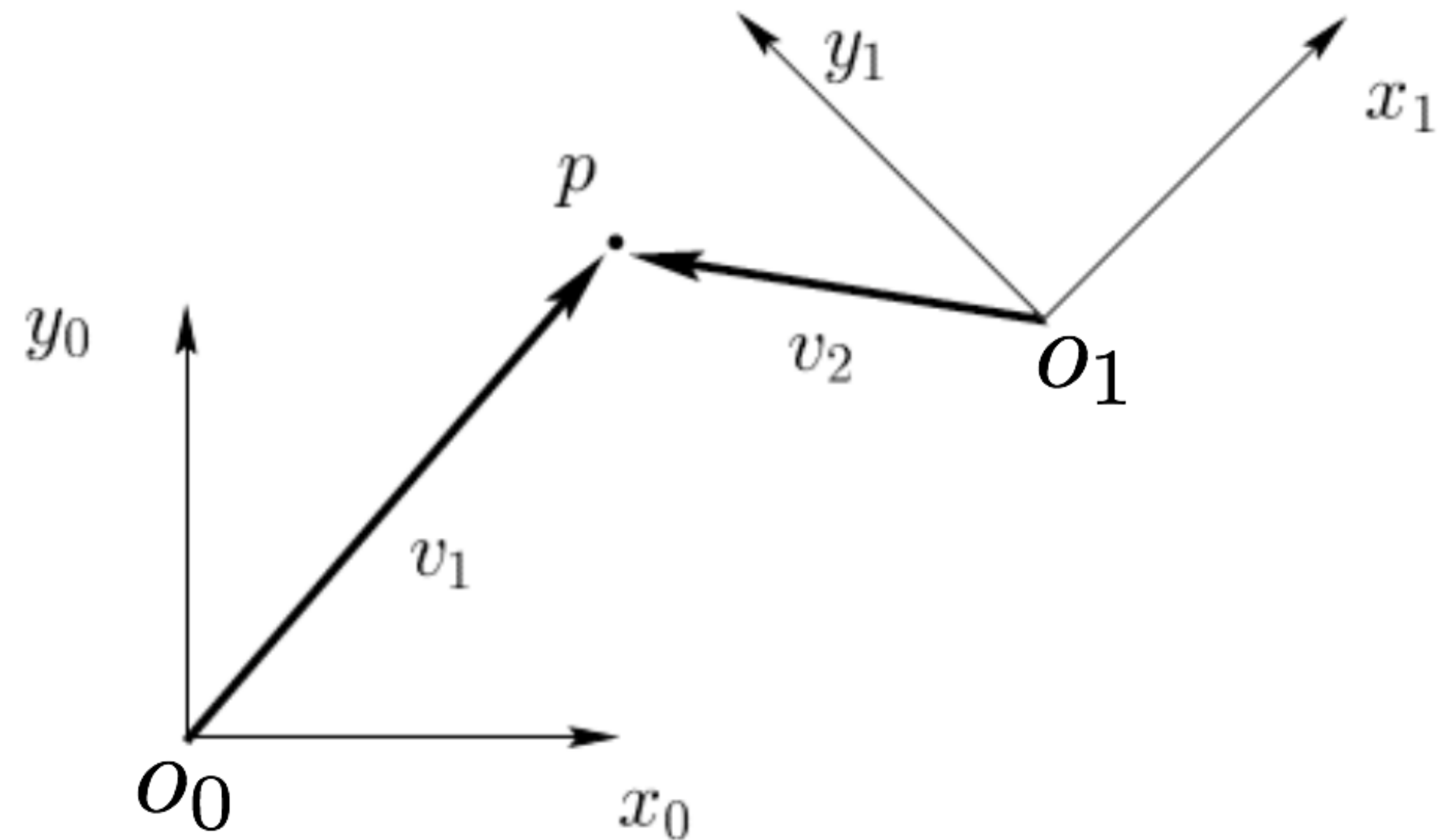
How many
solutions?



Coordinate Spaces as Systems of Linear Equations

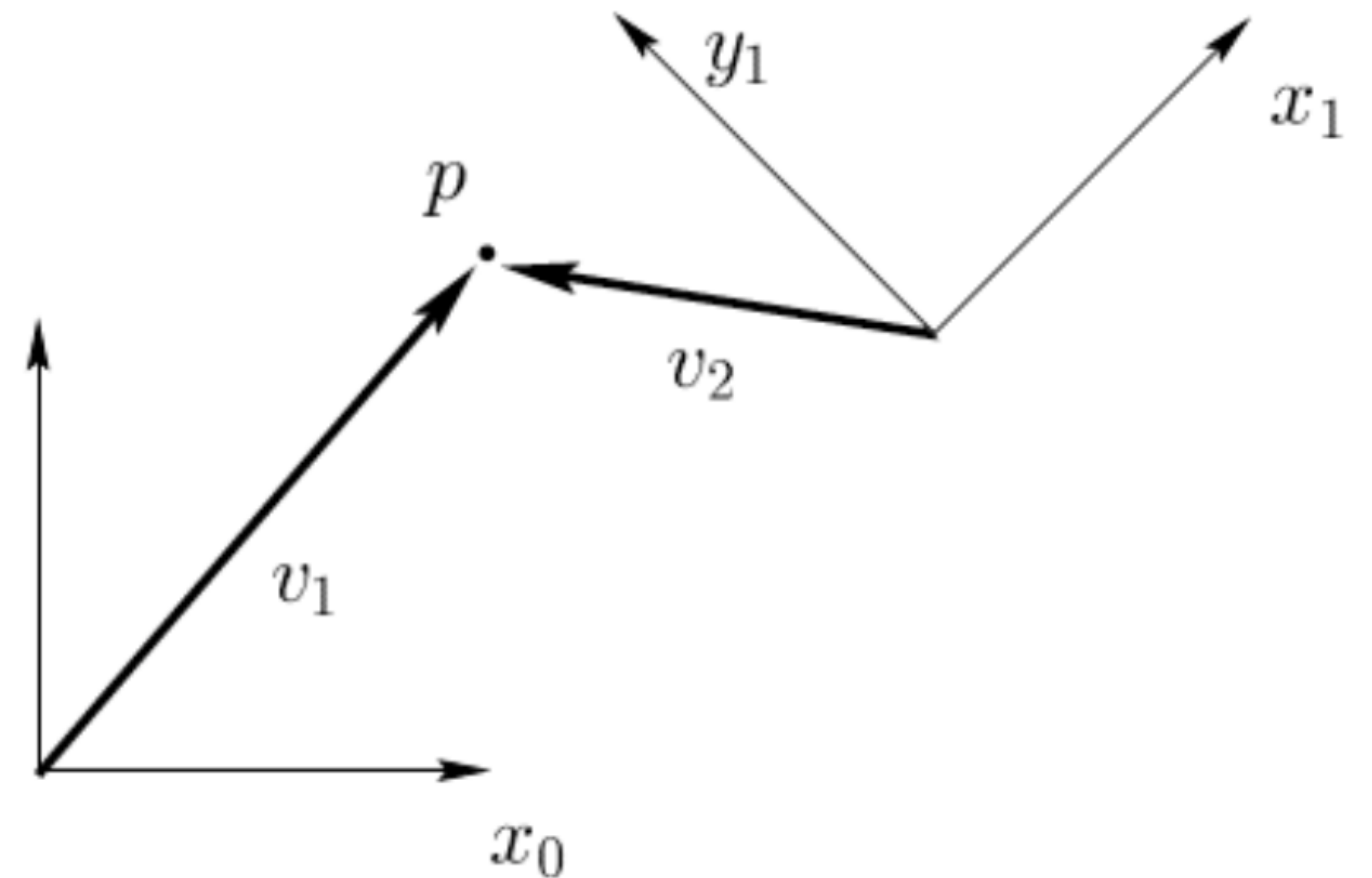
Coordinate Spaces (2D)

- Two coordinate frames $o_0x_0y_0$ and $o_1x_1y_1$, and a point p .
- The location of point p can be described with respect to either coordinate frame: $p^0 = [5, 6]^T$ and $p^1 = [-2.8, 4.2]^T$.
- The vector v_1 is direction and magnitude from o_0 to p , and v_2 is from o_1 to p .



Coordinate Spaces (2D)

- Point p has a location.
- Vectors v_1 and v_2 have directions and magnitudes.
- $v_1^0 = [5, 6]^T$ vector 1 in frame 0
- $v_1^1 = [7.77, 0.8]^T$ vector 1 in frame 1
- $v_2^0 = [-5.1, 1]^T$ vector 2 in frame 0
- $v_2^1 = [-2.8, 4.2]^T$ vector 2 in frame 1

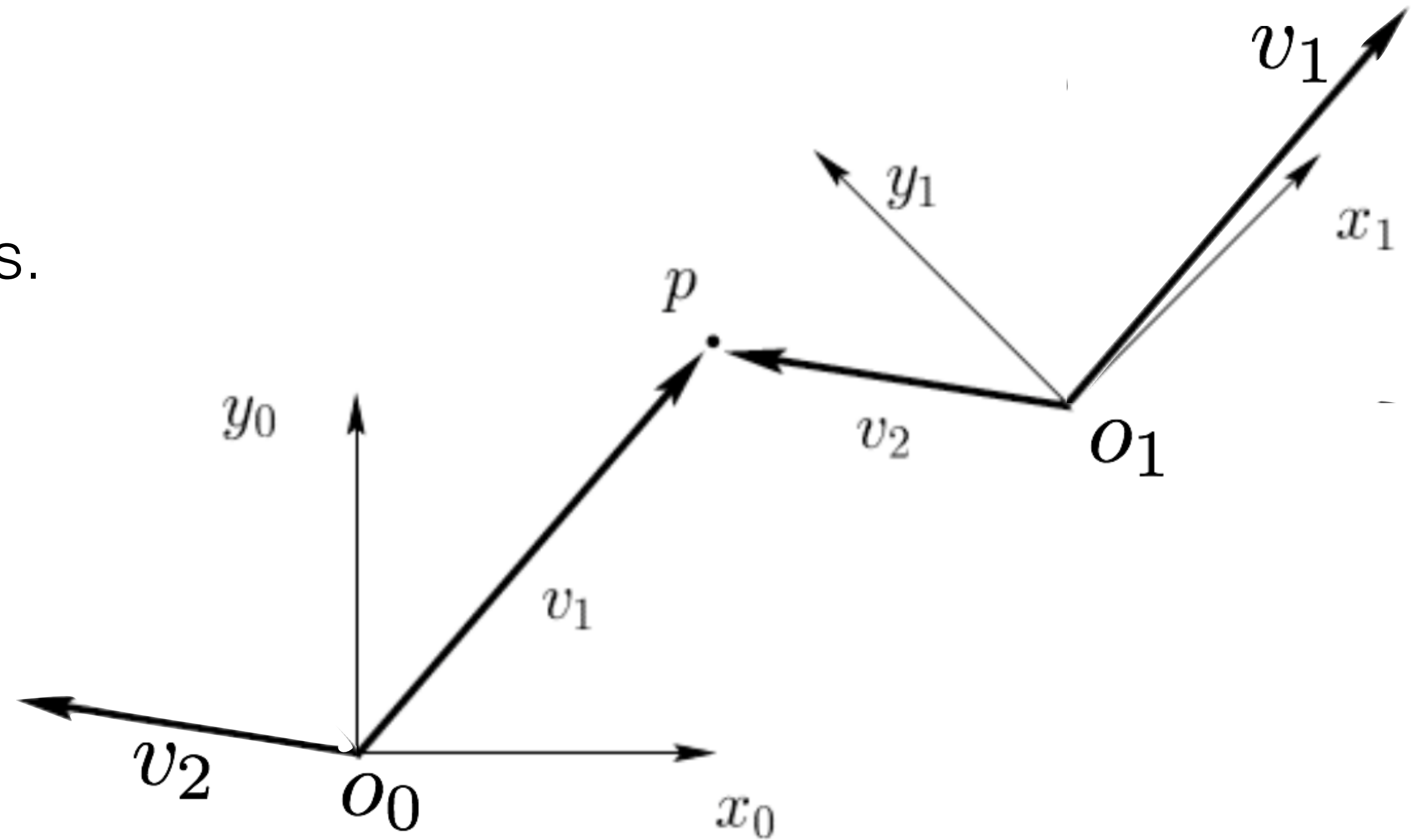


Note: Vectors can only be added when they are in the same coordinate frame.

Coordinate Spaces (2D)

- Point p has a location.
- Vectors v_1 and v_2 have directions and magnitudes.

- $v_1^0 = [5, 6]^T$
- $v_1^1 = [7.77, 0.8]^T$
- $v_2^0 = [-5.1, 1]^T$
- $v_2^1 = [-2.8, 4.2]^T$



Note: Vectors can only be added when they are in the same coordinate frame.

Vectors and Matrices

N-dimensional vector

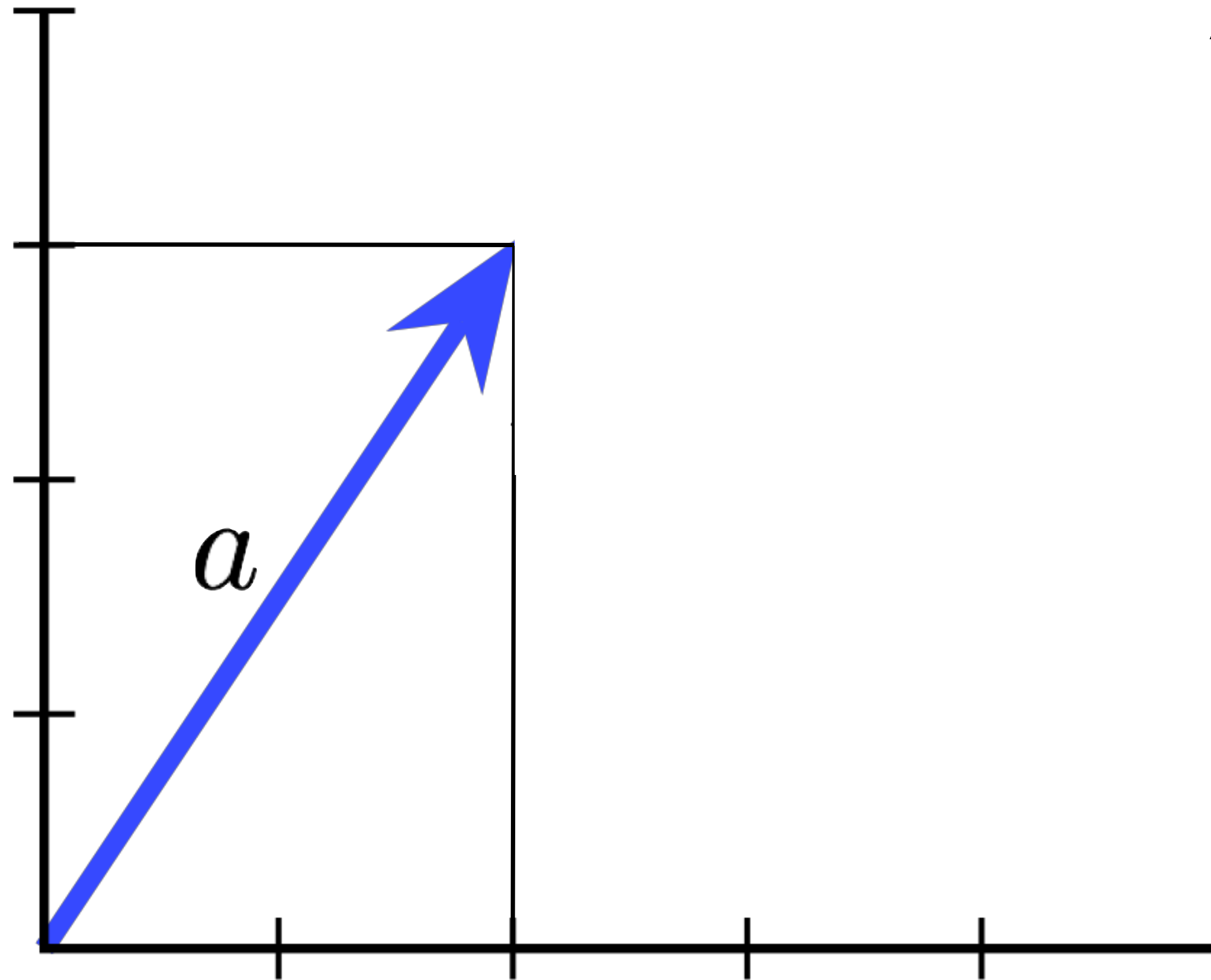
$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

M-by-N matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

2D Vector

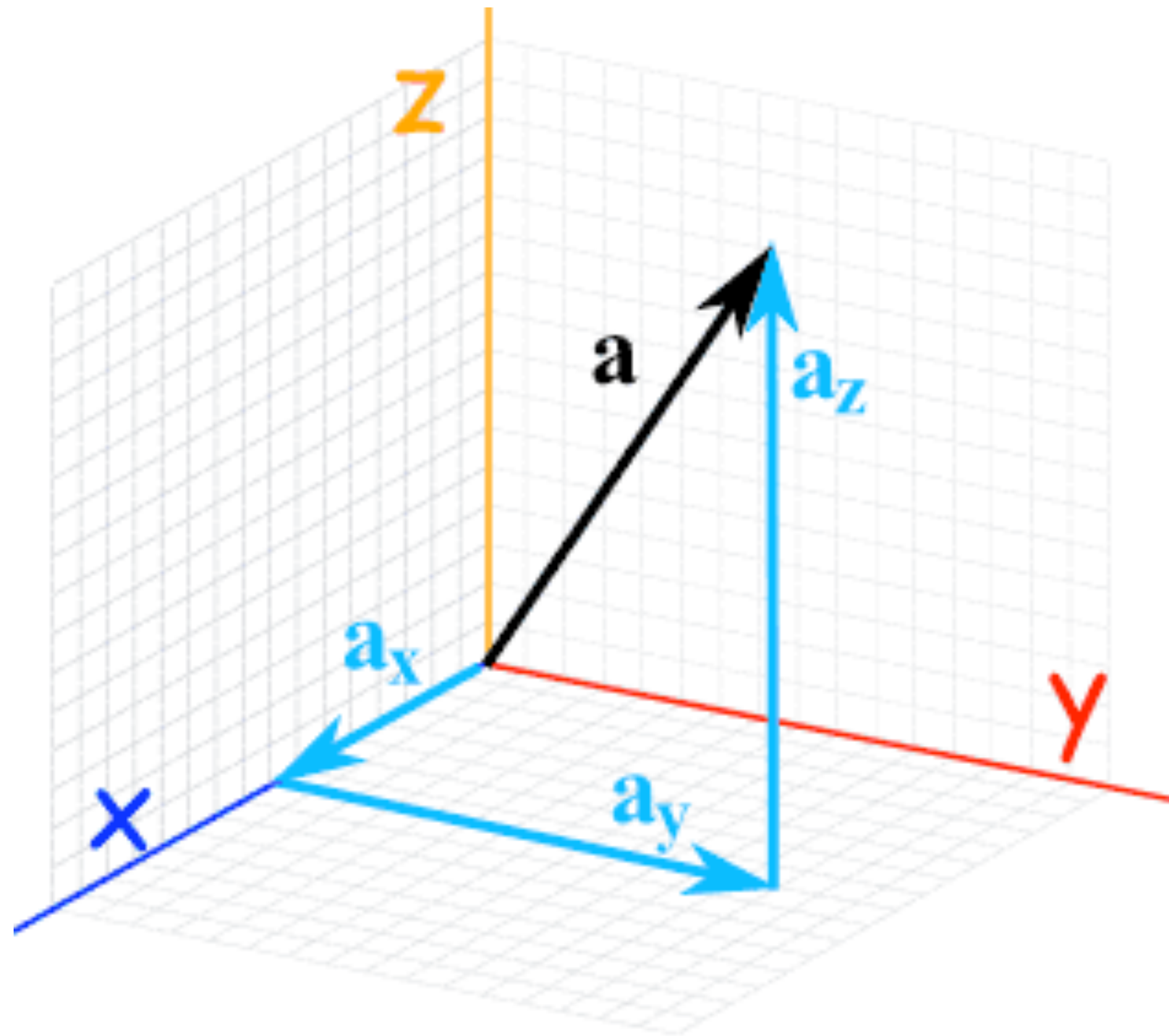
A vector is a motion in space



$$a = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

```
var a = [ [2],  
          [3] ];
```

3D Vector



$$a = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

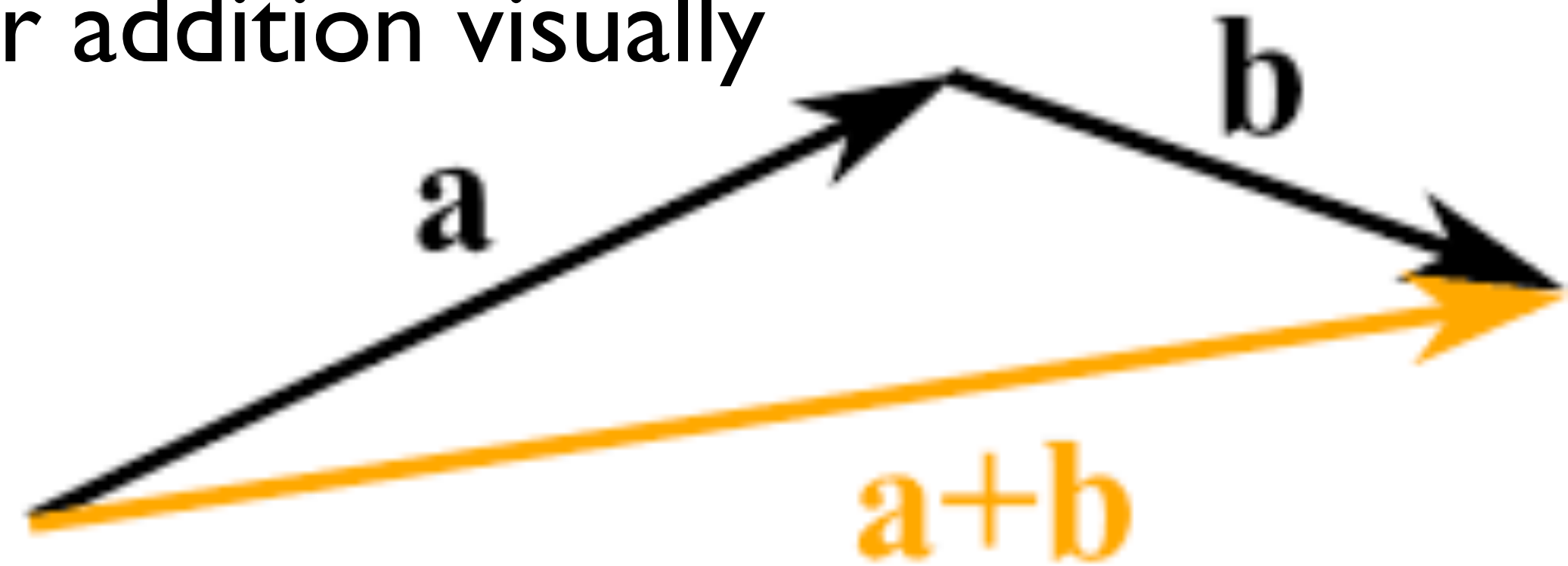
```
var a = [ [ax],  
          [ay],  
          [az] ];
```

Vector Addition and Subtraction

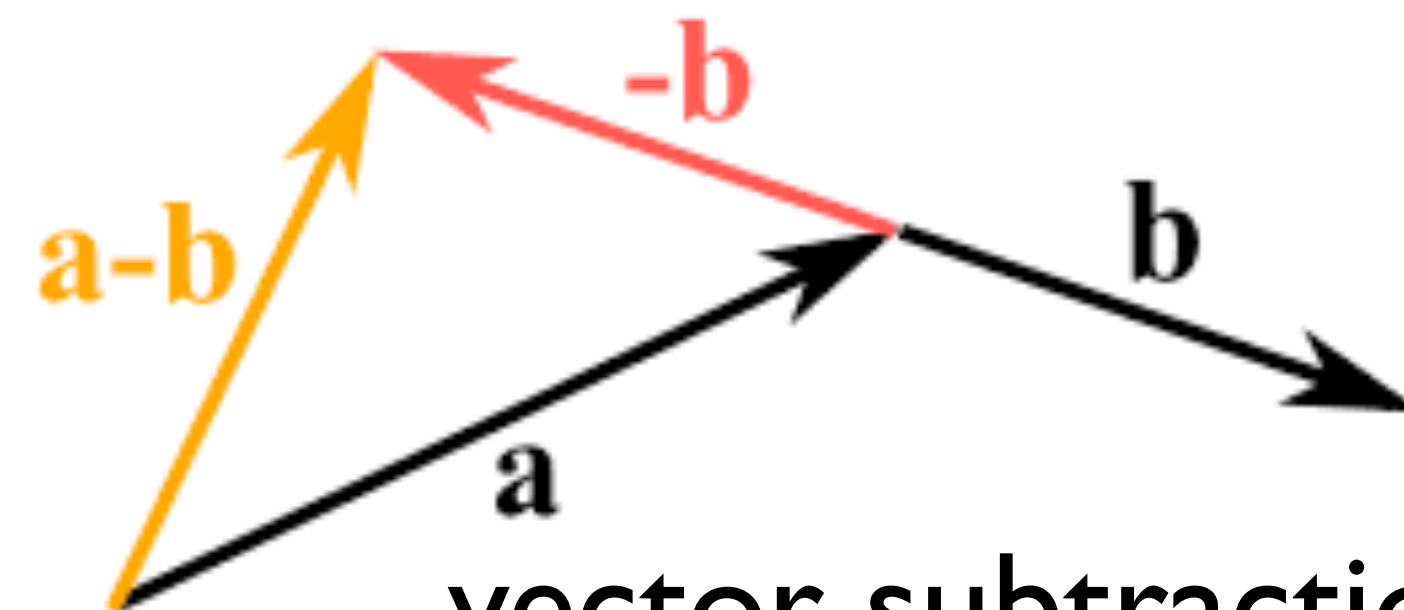
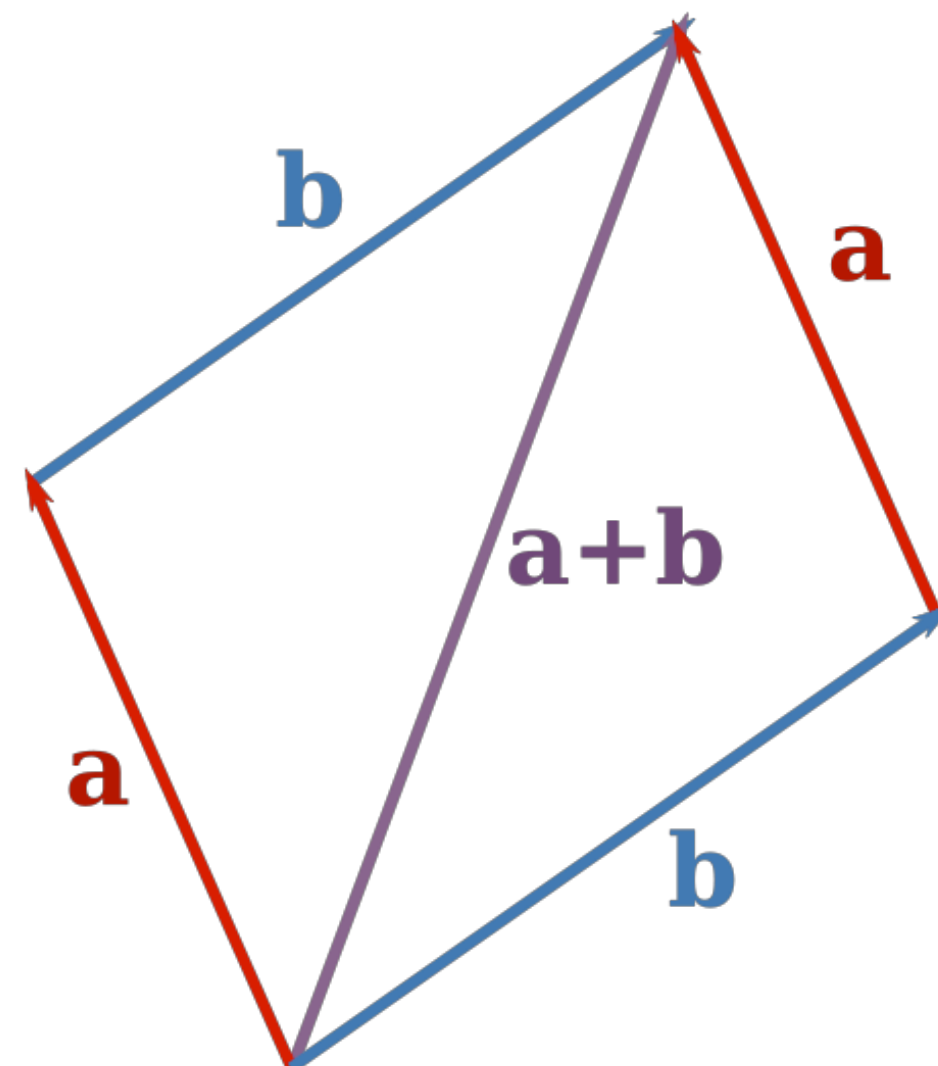
$$a + b = \begin{bmatrix} a_x + b_x \\ a_y + b_y \\ a_z + b_z \end{bmatrix}$$

vector
result

vector addition visually



vector addition is
order independent



vector subtraction is addition
with negated vector

Magnitude and Unit Vector

The magnitude of a vector is the square root of the sum of squares of its components

$$\|a\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

A unit vector has a magnitude of one.
Normalization scales a vector to unit length.

$$\hat{a} = \frac{a}{\|a\|}$$

A vector can be multiplied by a scalar

$$sa = \begin{bmatrix} sa_x \\ sa_y \\ sa_z \end{bmatrix}$$

scalar
result

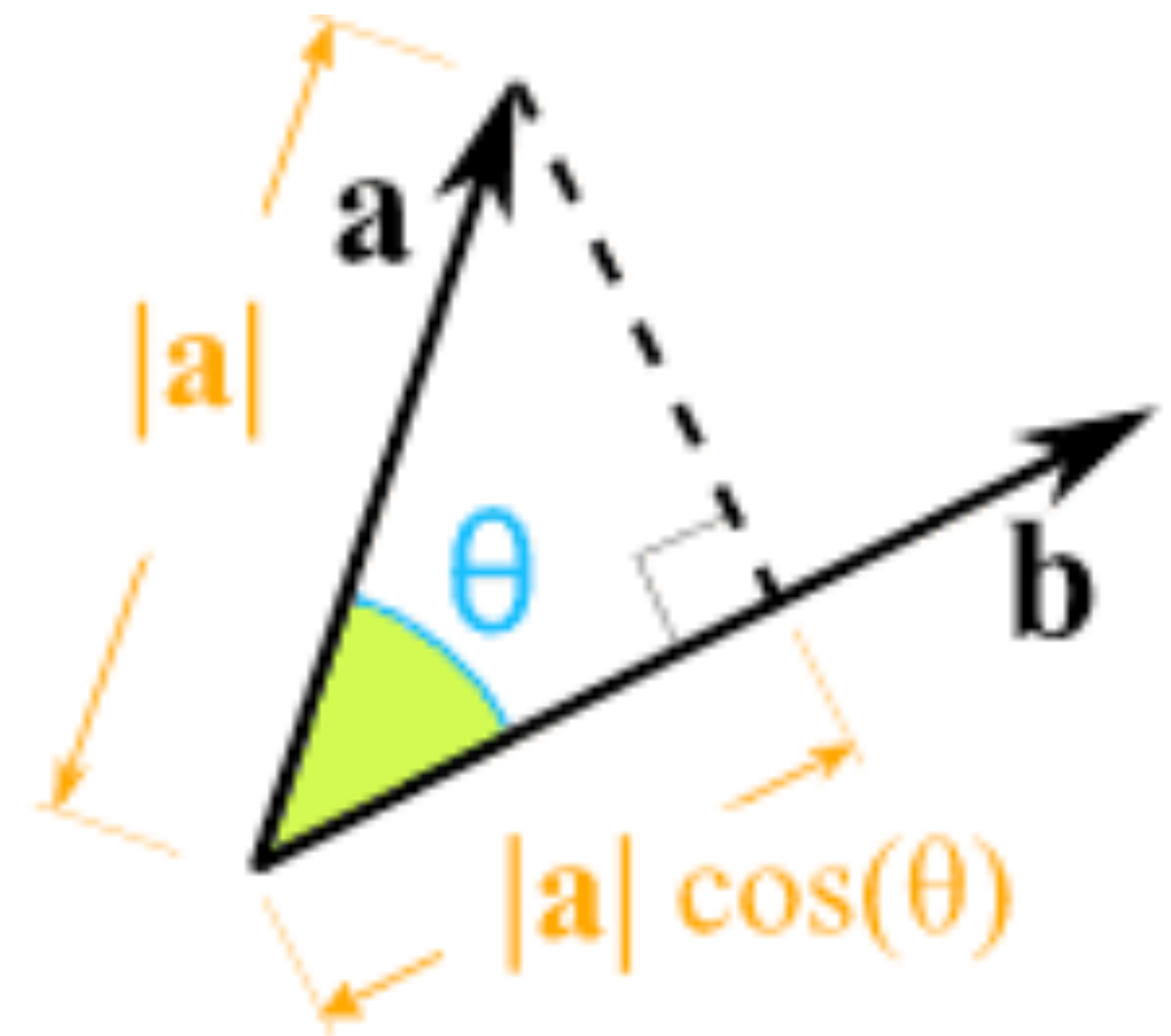
Dot Product

↓

$$\begin{aligned} a \bullet b &= a_x b_x + a_y b_y + a_z b_z \\ &= \|a\| \|b\| \cos(\theta) \end{aligned}$$

Measures the similarity in direction of
two vectors

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 2 * 3 + 1 * 2 = 8$$



Projections

Dot products related to projections onto vectors.

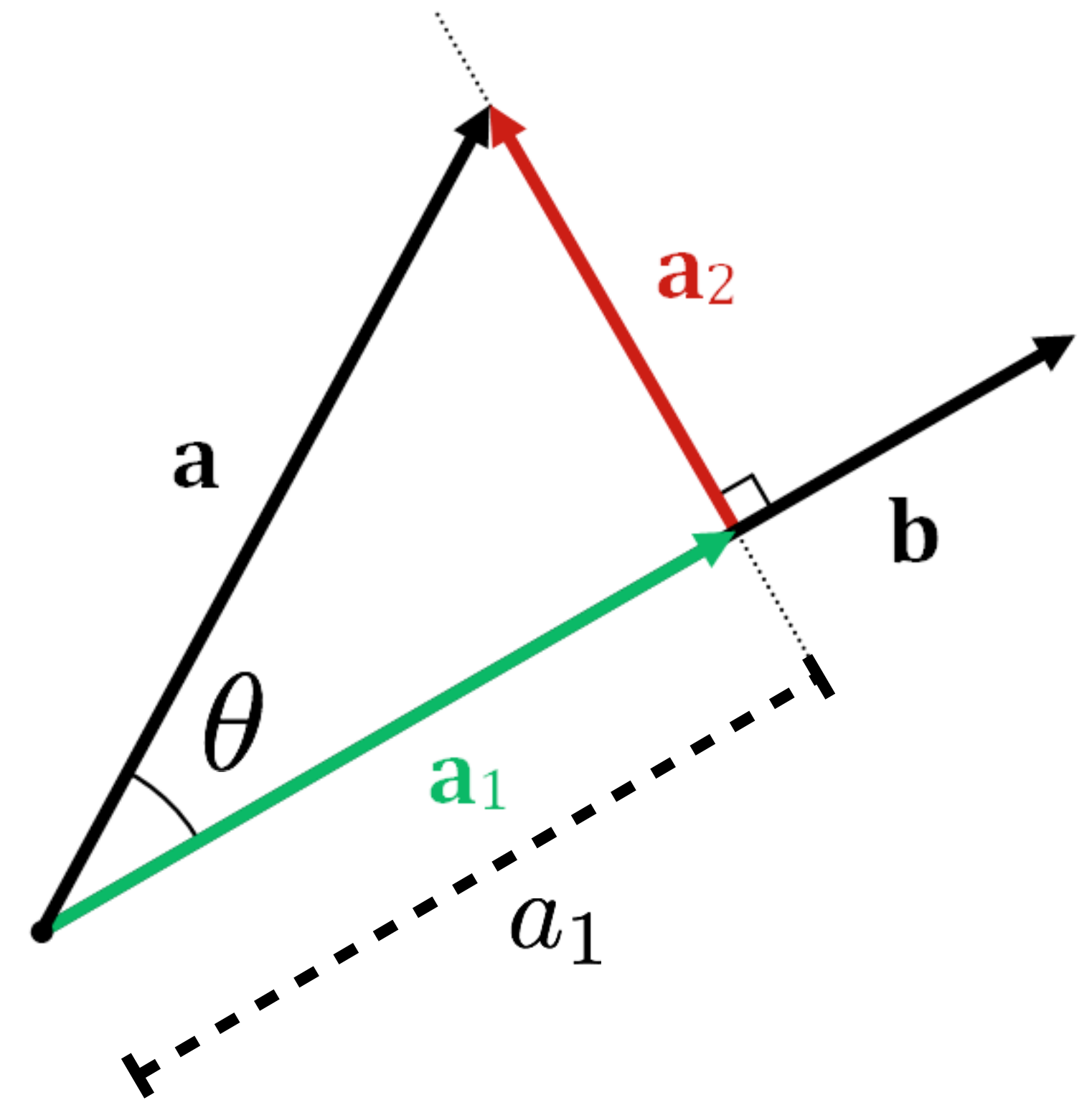
Scalar projection of one vector onto another

$$a_1 = |\mathbf{a}| \cos \theta = \mathbf{a} \cdot \hat{\mathbf{b}} = \mathbf{a} \cdot \frac{\mathbf{b}}{|\mathbf{b}|}$$

Vector projection

$$\mathbf{a}_1 = a_1 \hat{\mathbf{b}}$$

$\hat{\mathbf{b}}$ is unit length



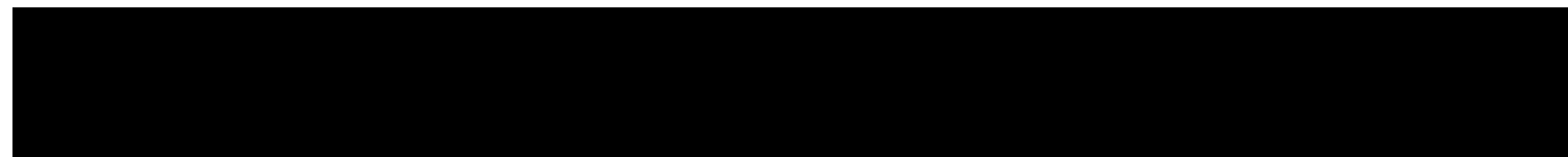
Checkpoint

- What is the dot product of a vector with itself?

- What is the dot product of two orthogonal vectors?

Checkpoint

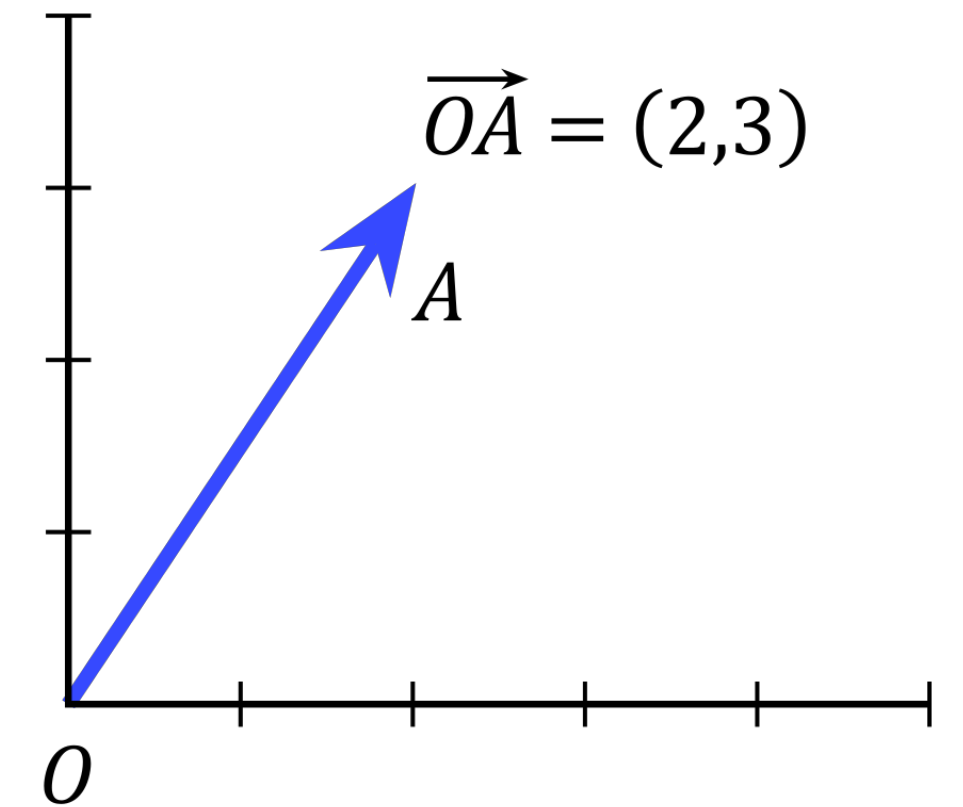
- What is the dot product of a vector with itself?
 - the square of the vector magnitude
- What is the dot product of two orthogonal vectors?



Checkpoint

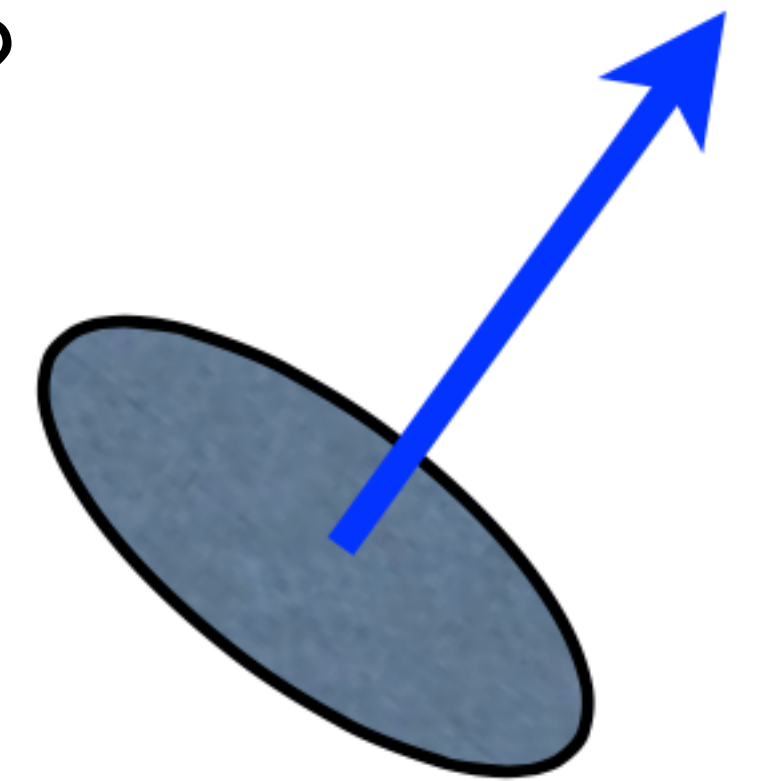
- What is the dot product of a vector with itself?
 - the square of the vector magnitude
- What is the dot product of two orthogonal vectors?
 - 0

Checkpoint

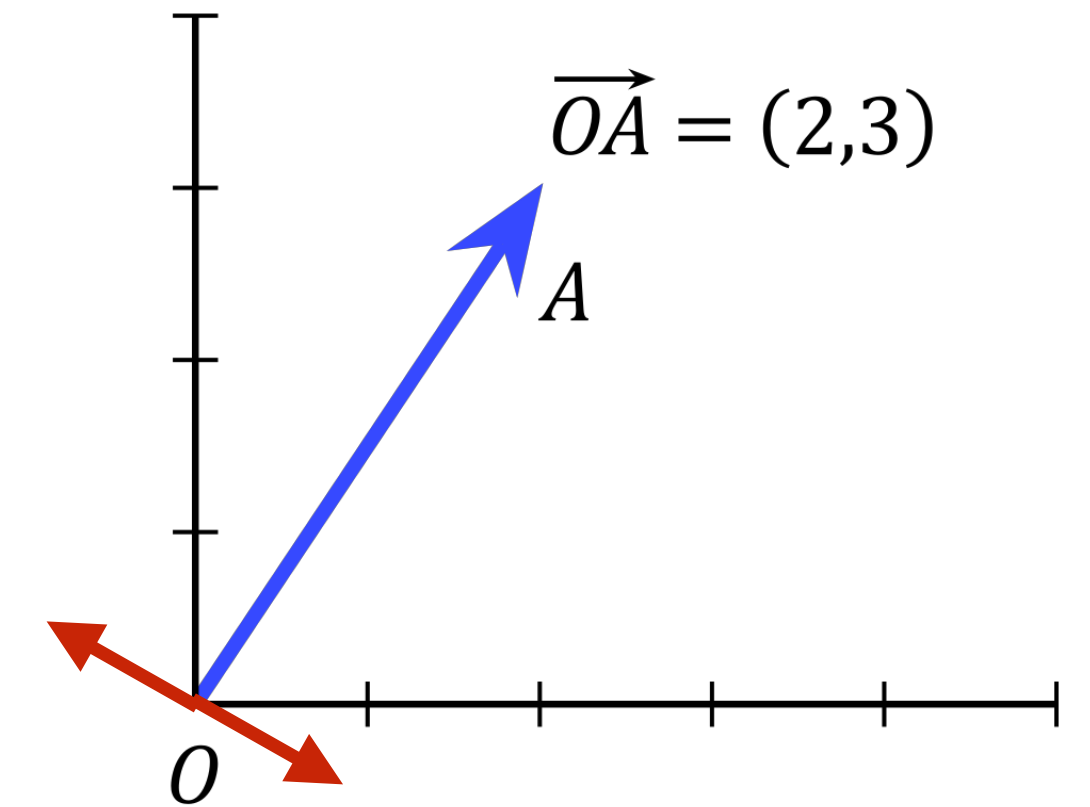


- How many unit vectors are perpendicular to a 2D vector?

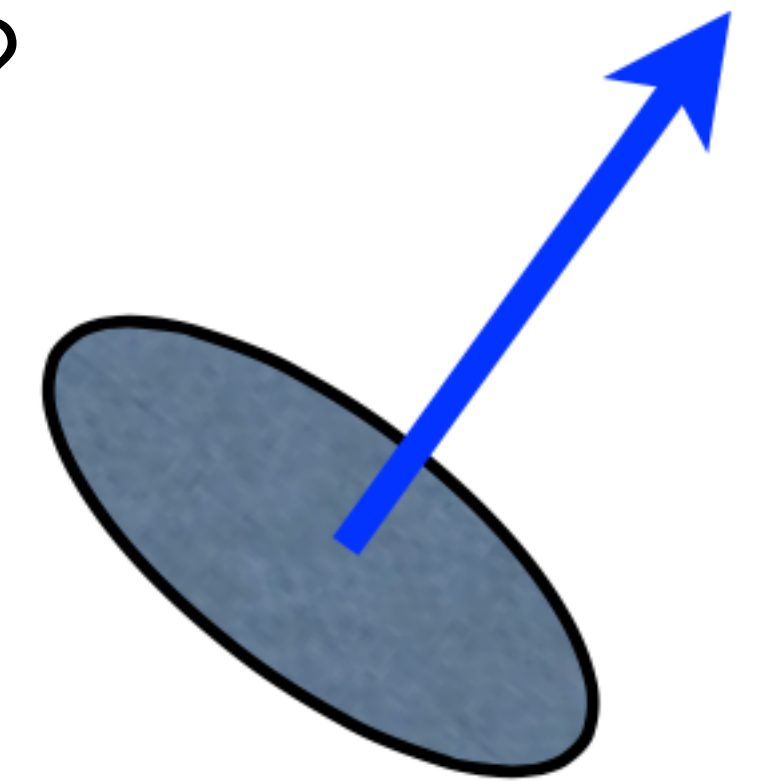
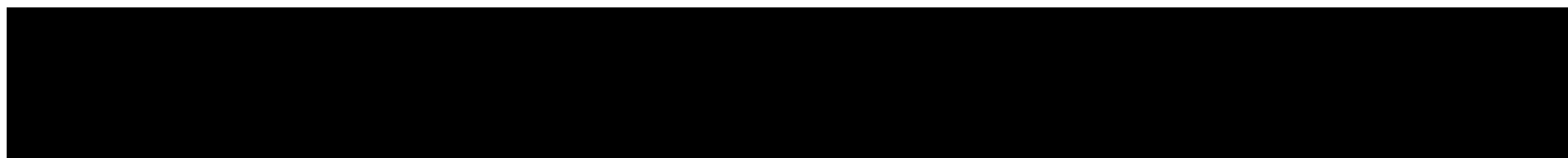
- How many unit vectors are perpendicular to a 3D vector?



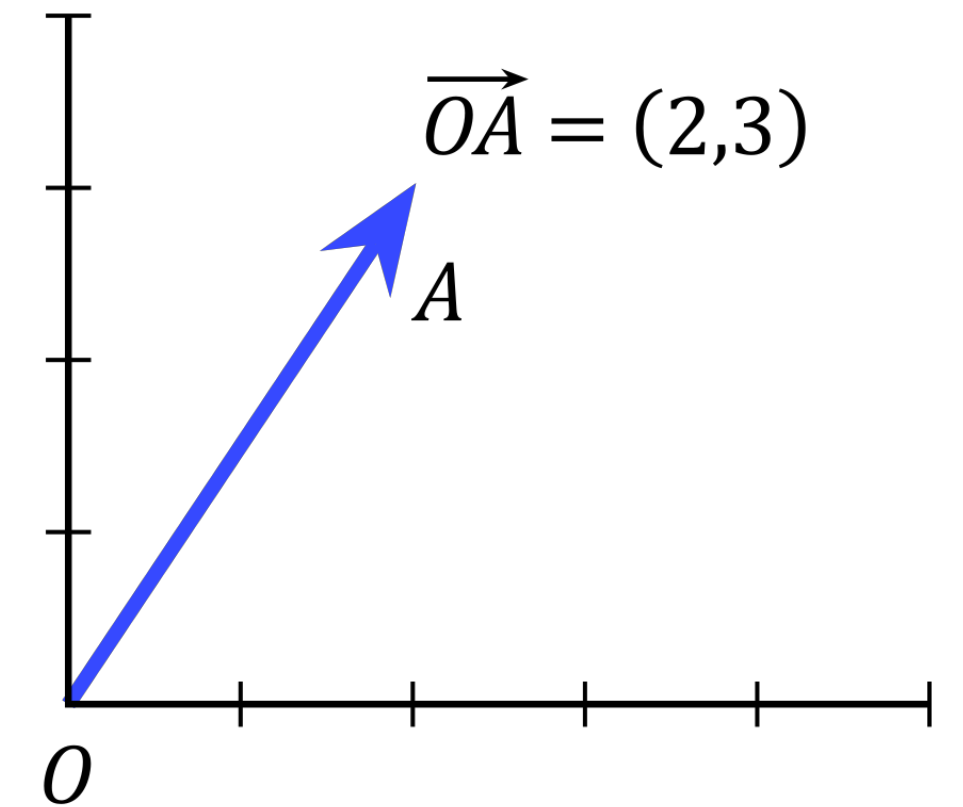
Checkpoint



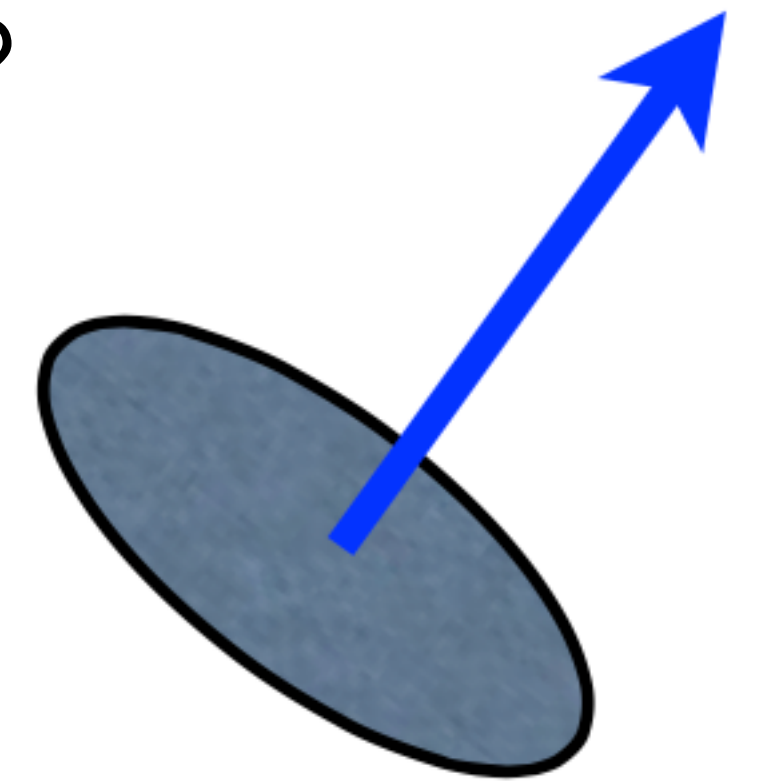
- How many unit vectors are perpendicular to a 2D vector?
 - 2 (positive and negative)
- How many unit vectors are perpendicular to a 3D vector?



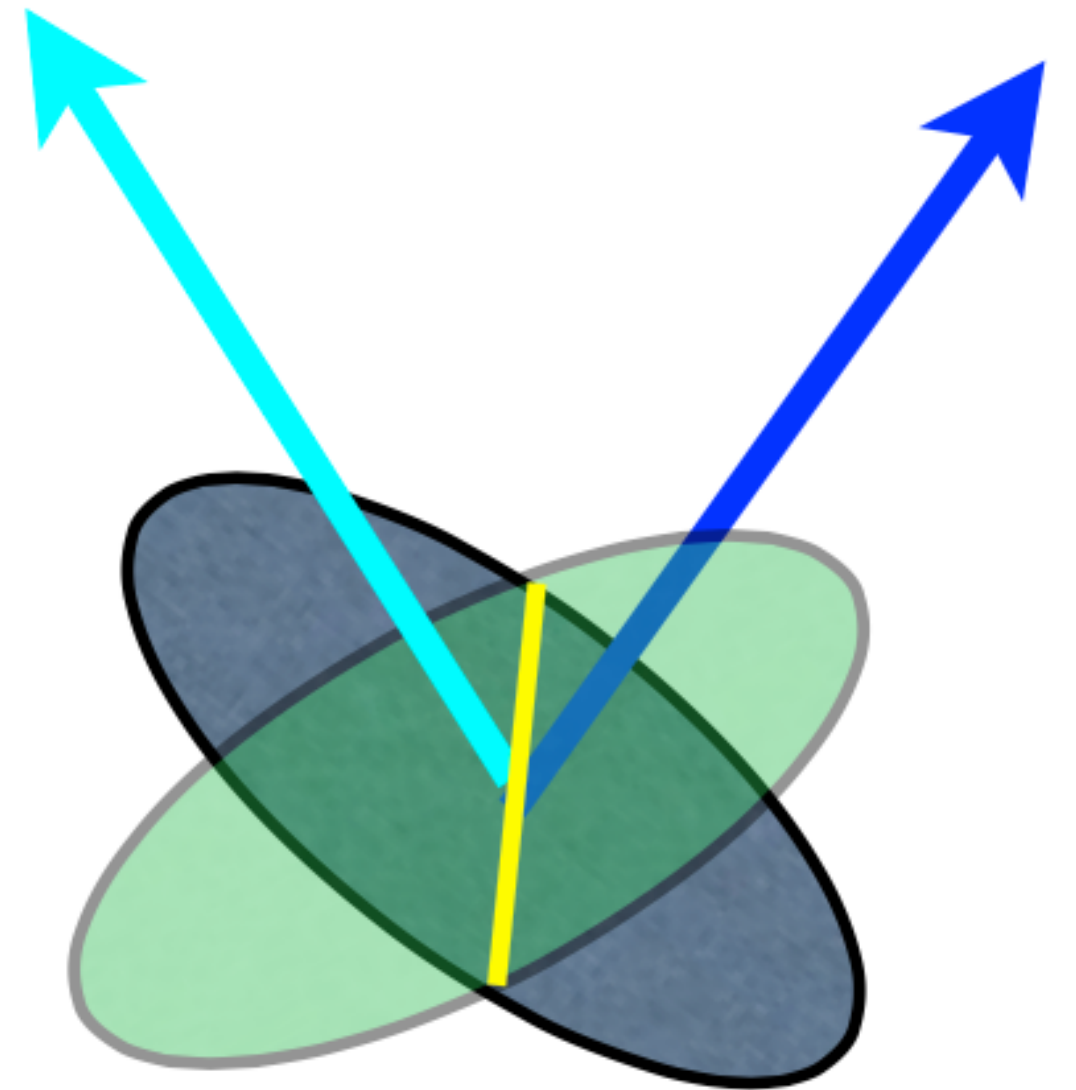
Checkpoint



- How many unit vectors are perpendicular to a 2D vector?
 - 2 (positive and negative)
- How many unit vectors are perpendicular to a 3D vector?
 - Infinite and lie in plane



Given two vectors, how to compute
a vector orthogonal to both?



Cross Product

$$c_x = a_y b_z - a_z b_y$$

$$c_y = a_z b_x - a_x b_z$$

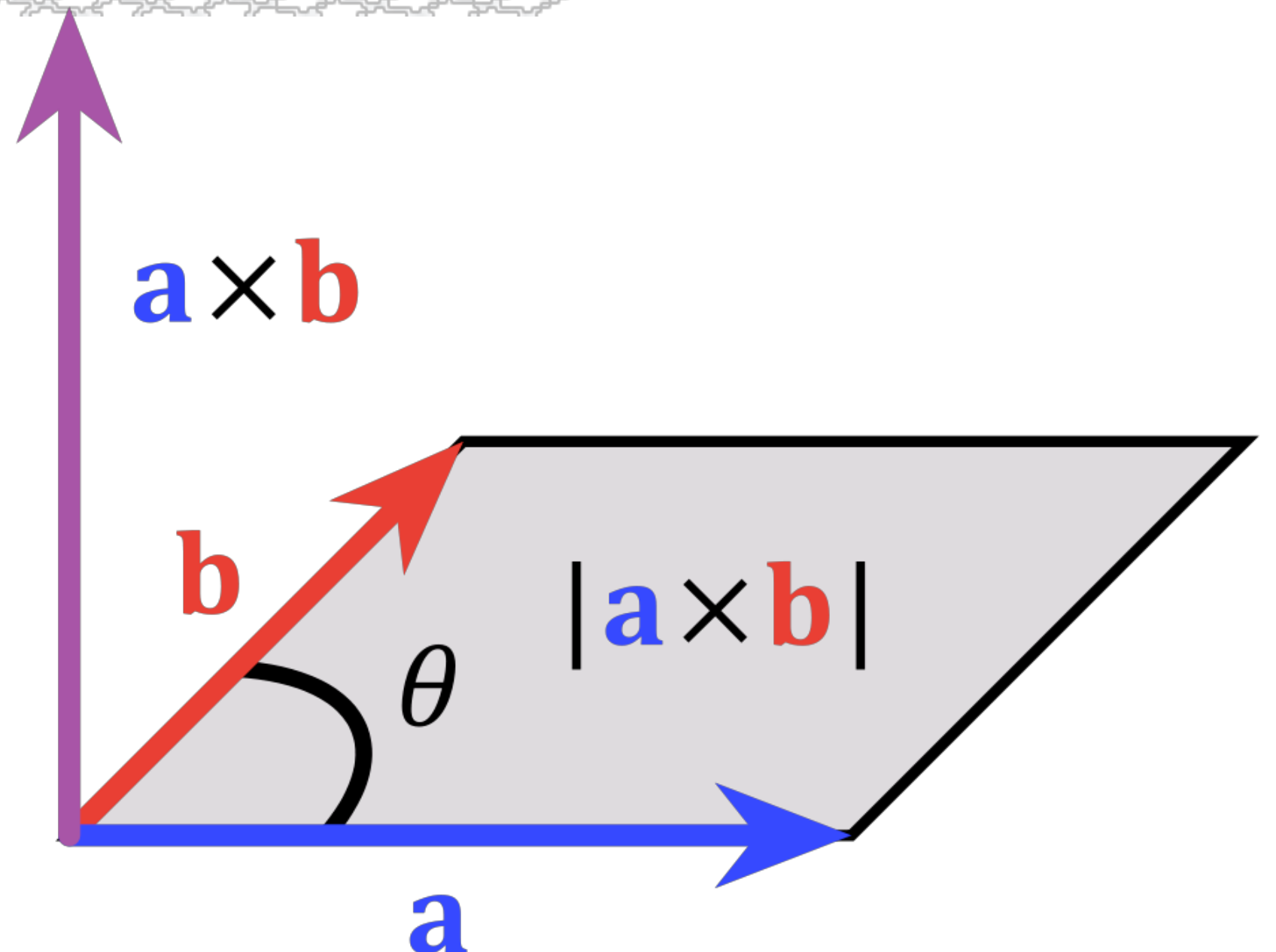
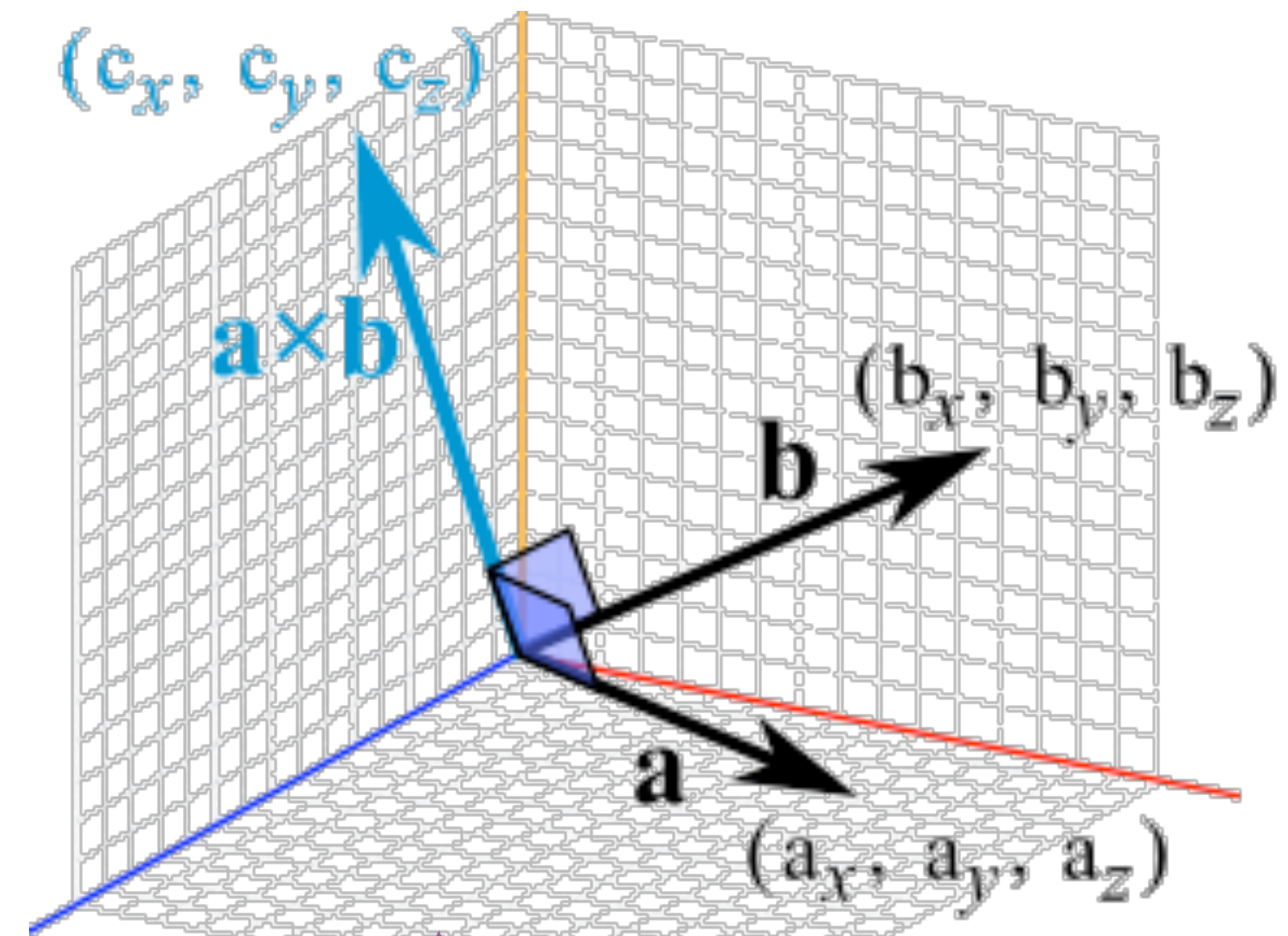
$$c_z = a_x b_y - a_y b_x$$

Results in new vector c orthogonal to both original vectors a and b

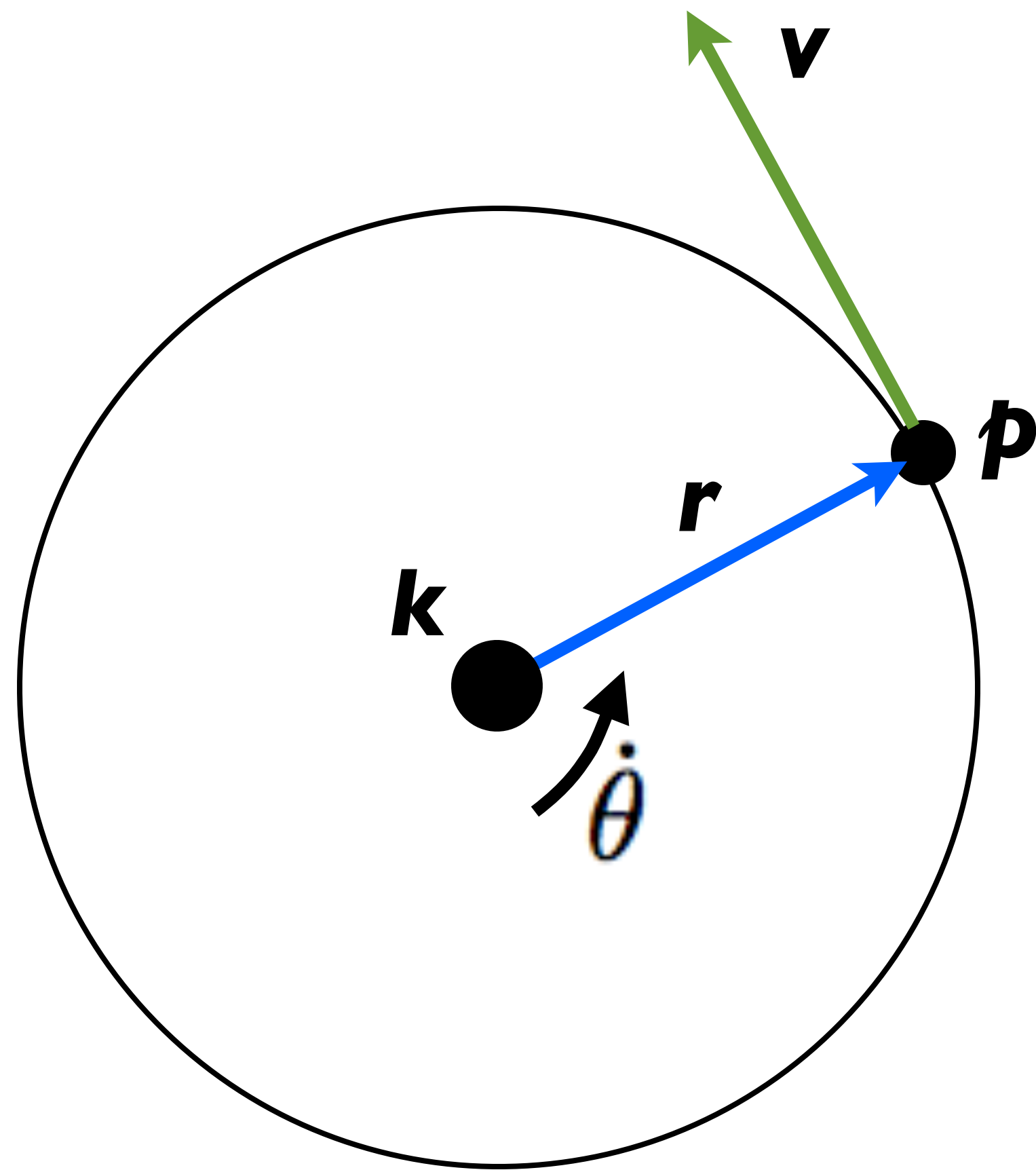
Length of vector c is equal to area of parallelogram formed by a and b

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta$$

Assumes $\mathbf{a}$ and $\mathbf{b}$ are in same frame



Relating linear and angular velocity



$$v = \dot{\theta} k \times r$$

vector to a point p

linear velocity of point p

angular velocity for rotation of p about vector k

Matrices

- A Matrix is a rectangular array of numbers

```
var mat = [  
  [1, 0, 0, 0],  
  [0, 1, 0, 0],  
  [0, 0, 1, 0],  
  [0, 0, 0, 1] ];
```

What is this
matrix?

Matrix-vector multiplication

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j \\ k \\ l \end{bmatrix} = \begin{bmatrix} aj + bk + cl \\ dj + ek + fl \\ gj + hk + il \end{bmatrix}$$

For example

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} =$$



For example

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 10 \\ -1 \end{bmatrix}$$

Matrix-vector multiplication

(two interpretations)

1) **Row story:** dot product of each matrix row

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j \\ k \\ l \end{bmatrix} = \begin{bmatrix} aj + bk + cl \\ dj + ek + fl \\ gj + hk + il \end{bmatrix}$$

2) **Column story:** linear combination of matrix columns

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j \\ k \\ l \end{bmatrix} = \begin{bmatrix} aj + bk + cl \\ dj + ek + fl \\ gj + hk + il \end{bmatrix} = \begin{bmatrix} a \\ d \\ g \end{bmatrix} j + \begin{bmatrix} b \\ e \\ h \end{bmatrix} k + \begin{bmatrix} c \\ f \\ i \end{bmatrix} l$$

Revisiting the cross product: Skew-symmetric matrices

A given 3D vector $\mathbf{a} = (a_1 \ a_2 \ a_3)^T$

can be expressed as a skew-symmetric matrix

$$[\mathbf{a}]_{\times} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

such that the cross product with another vector is a matrix multiplication

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_{\times} \mathbf{b}$$

Linear Systems

We can use a variable instead of a vector, which gives us a linear system.

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} x = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Enabling the general form: **$A\mathbf{x} = \mathbf{b}$**

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

Matrices

- A Matrix is a rectangular array of numbers

```
var mat = [  
  [1, 0, 0, 0],  
  [0, 1, 0, 0],  
  [0, 0, 1, 0],  
  [0, 0, 0, 1] ];
```

```
var mat = [  
  [1, 0, 0, tx],  
  [0, 1, 0, ty],  
  [0, 0, 1, tz],  
  [0, 0, 0, 1] ];
```

What is this
matrix?

Translation matrix example

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} =$$



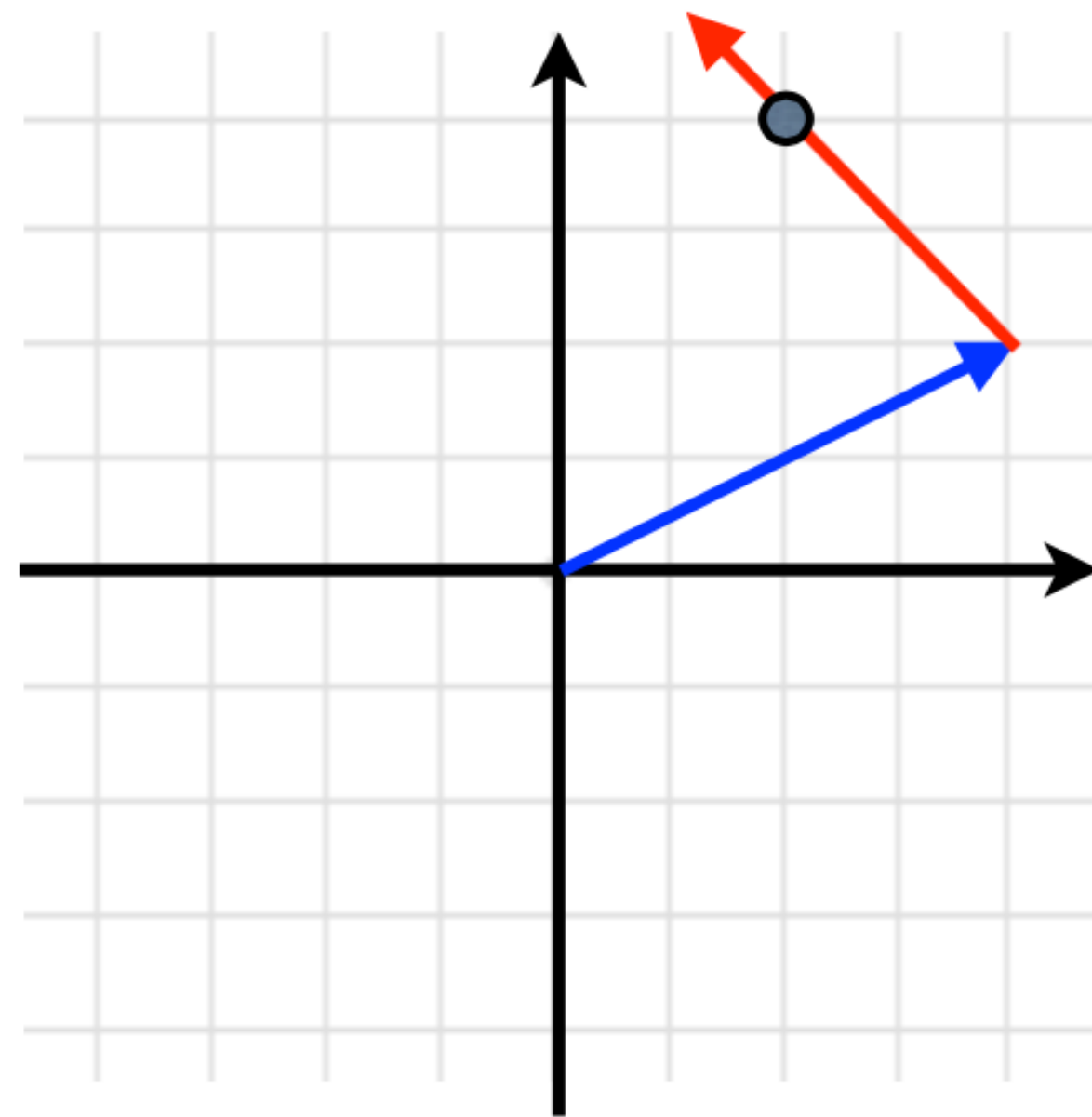
↑
jsmat for Assignment 3

Translation matrix example

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ z + t_z \\ 1 \end{bmatrix}$$


jsmat for Assignment 3

Matrix Geometry: Column Story



- Each column can be interpreted as a vector
 - How far do we go in each direction?

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} x = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} -1 \\ 1 \end{bmatrix} x_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Matrix Multiplication

- Scalar Multiplication

$$\lambda \mathbf{A} = \lambda \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \end{pmatrix} = \begin{pmatrix} \lambda A_{11} & \lambda A_{12} & \cdots & \lambda A_{1m} \\ \lambda A_{21} & \lambda A_{22} & \cdots & \lambda A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda A_{n1} & \lambda A_{n2} & \cdots & \lambda A_{nm} \end{pmatrix}.$$

- Multiplication of two matrices

$$(\mathbf{AB})_{ij} = \sum_{k=1}^m A_{ik} B_{kj}.$$

Each entry of product matrix $\mathbf{AB}$ is a dot product of a row of $\mathbf{A}$ with a column of $\mathbf{B}$

Matrix multiplication

Finger sweeping rule should be second nature!

- Left finger sweeps left to right
- Right finger sweeps top to bottom

Diagram illustrating matrix multiplication:

Matrix A (3x4 matrix):

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{bmatrix}$$

Matrix B (4x5 matrix):

$$\begin{bmatrix} \cdot & \cdot & \cdot & \mathbf{a} & \cdot \\ \cdot & \cdot & \cdot & \mathbf{b} & \cdot \\ \cdot & \cdot & \cdot & \mathbf{c} & \cdot \\ \cdot & \cdot & \cdot & \mathbf{d} & \cdot \end{bmatrix}$$

Matrix AB (3x5 matrix):

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & x_{3,4} & \cdot \end{bmatrix}$$

Labels:

- row 3 of A
- col 4 of B

Do this dot product for each row/column combination

$$\begin{aligned} x_{3,4} &= (\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}) \cdot (\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \\ &= \mathbf{1} \times \mathbf{a} + \mathbf{2} \times \mathbf{b} + \mathbf{3} \times \mathbf{c} + \mathbf{4} \times \mathbf{d} \end{aligned}$$

Matrix Multiplication Reminders

- Number of columns of A must match number of rows of B
- Multiplying a $(M \times K)$ matrix with a $(K \times N)$ matrix will produce an $(M \times N)$ matrix
- Matrix multiplication is not commutative: $AB \neq BA$

Example

"Dot Product"

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 \end{bmatrix}$$

$$(1, 2, 3) \cdot (7, 9, 11) = 1 \times 7 + 2 \times 9 + 3 \times 11 = 58$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \end{bmatrix}$$

$$(1, 2, 3) \cdot (8, 10, 12) = 1 \times 8 + 2 \times 10 + 3 \times 12 = 64$$

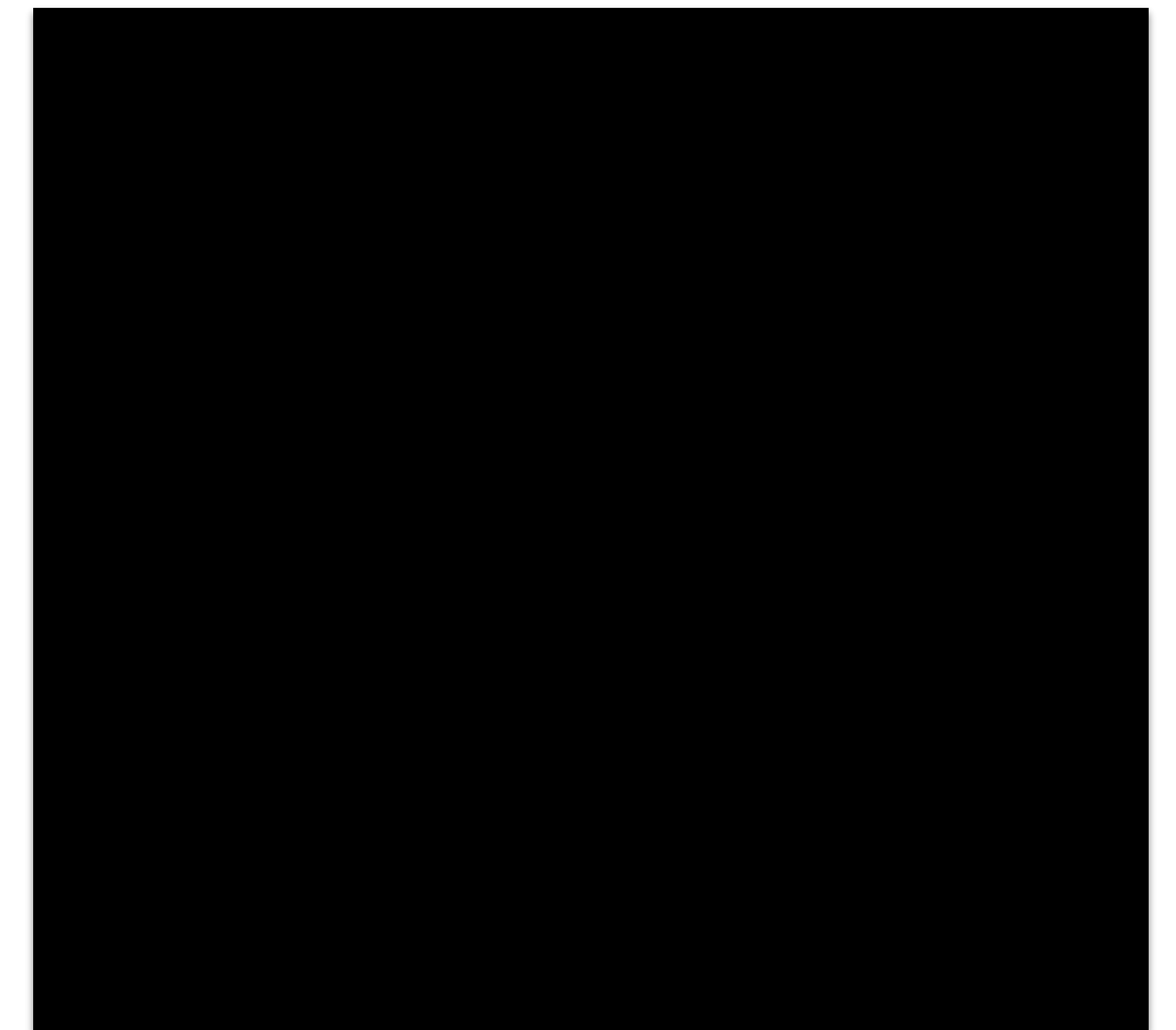
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

$$(4, 5, 6) \cdot (7, 9, 11) = 4 \times 7 + 5 \times 9 + 6 \times 11 = 139$$

$$(4, 5, 6) \cdot (8, 10, 12) = 4 \times 8 + 5 \times 10 + 6 \times 12 = 154$$

For example

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 3 \end{bmatrix} =$$

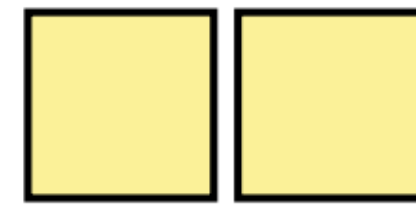


For example

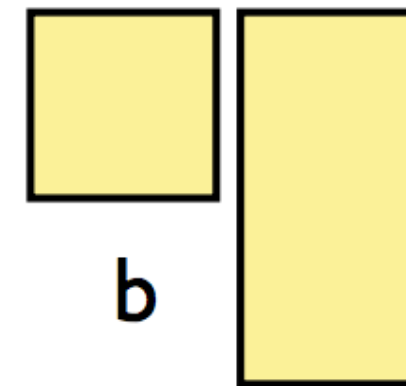
$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ 2 & 5 \end{bmatrix}$$

Checkpoint

- Which of the following matrix multiplications are valid?



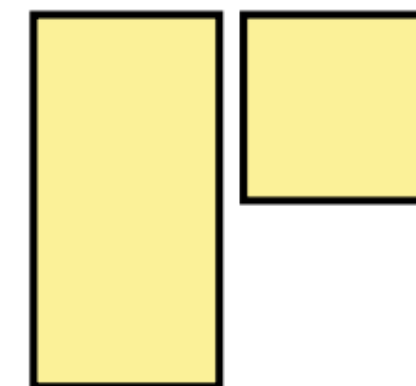
a



b



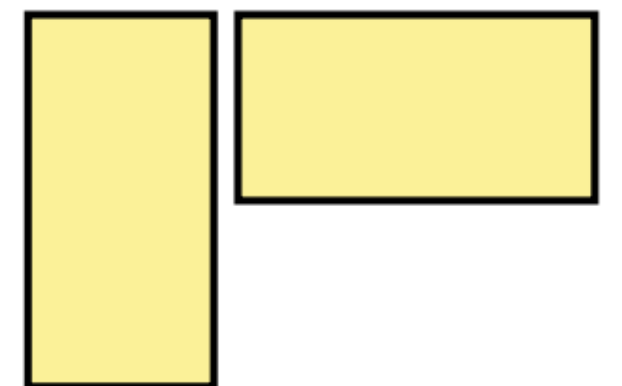
c



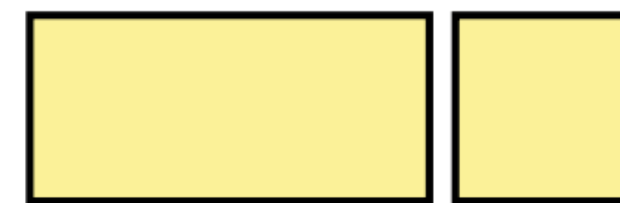
d



e



f



g



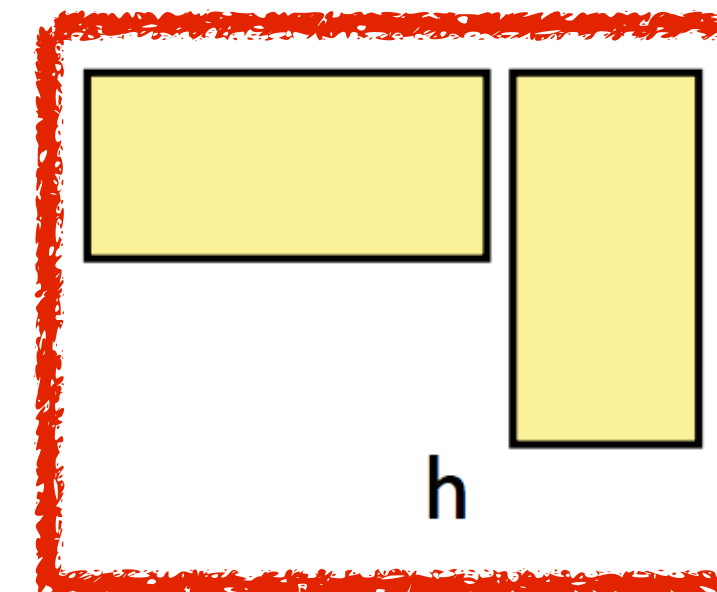
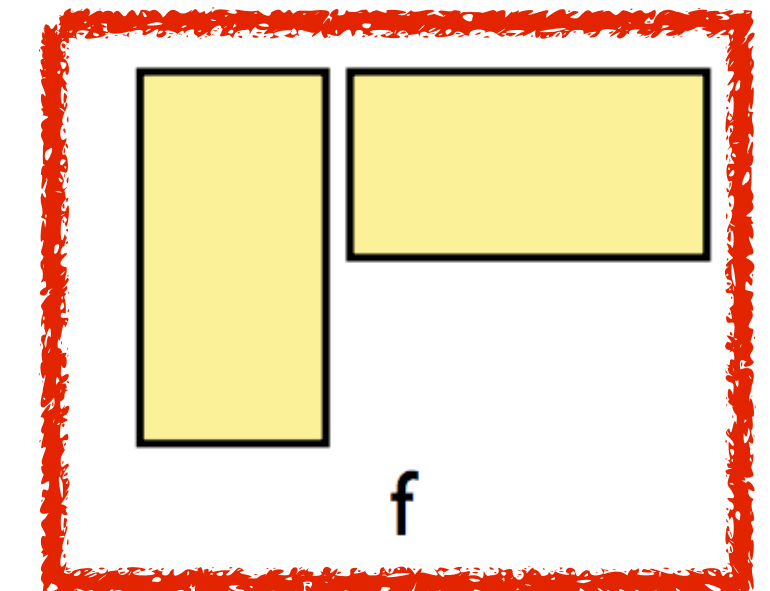
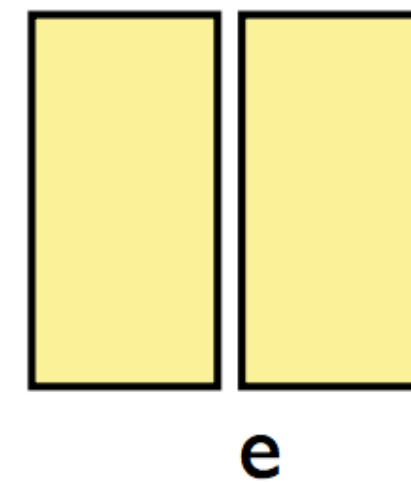
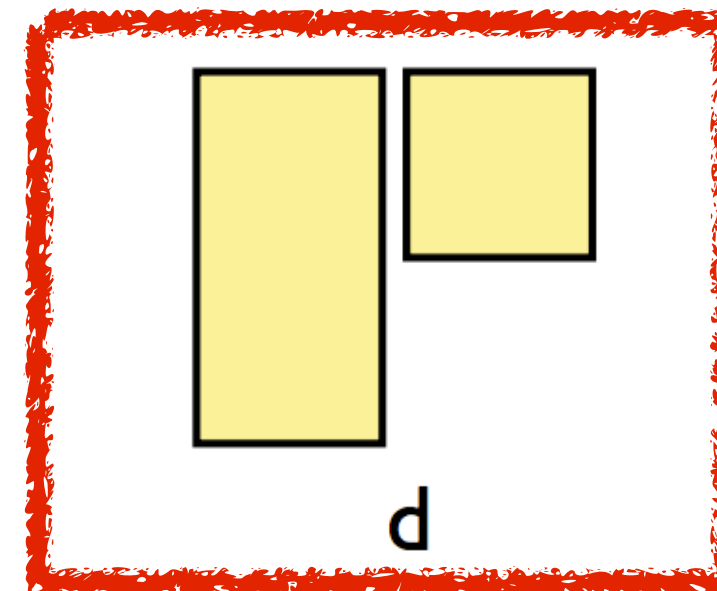
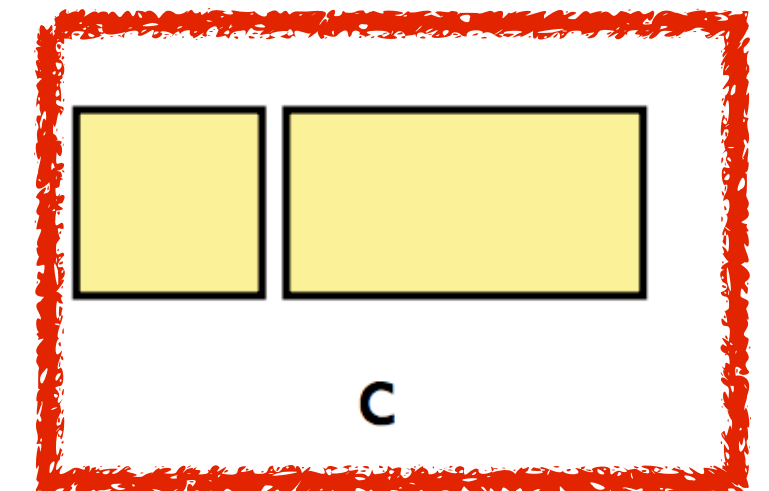
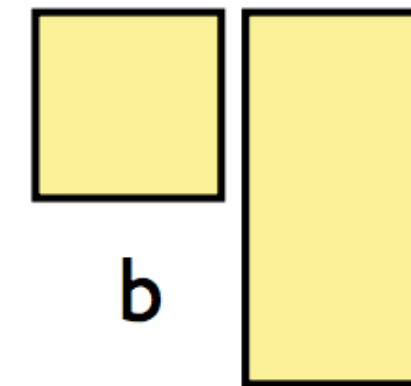
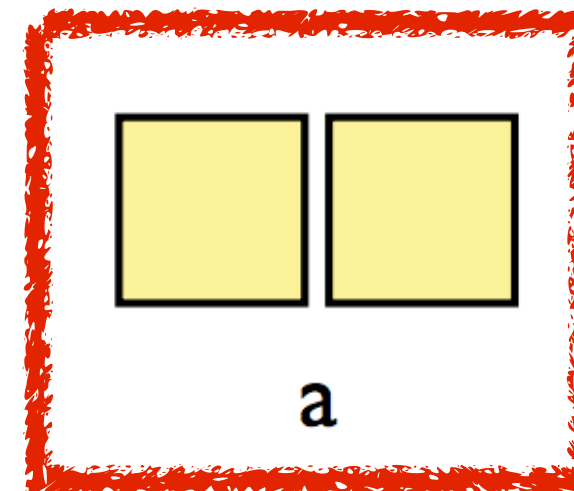
h



i

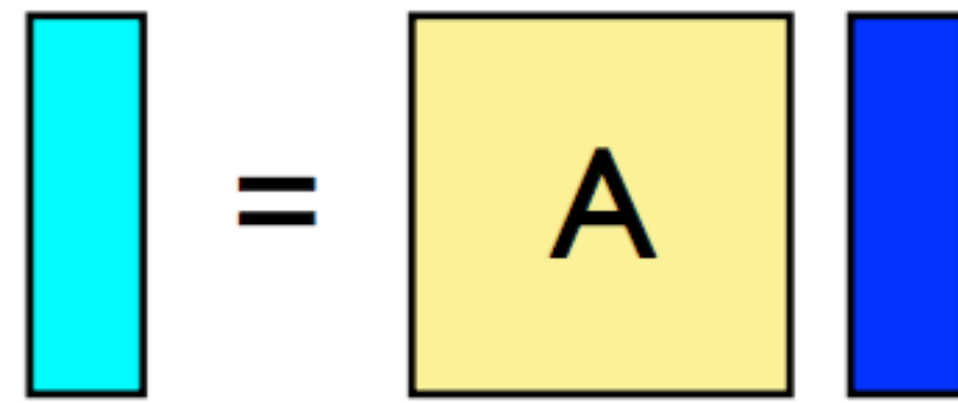
Checkpoint

- Which of the following matrix multiplications are valid?



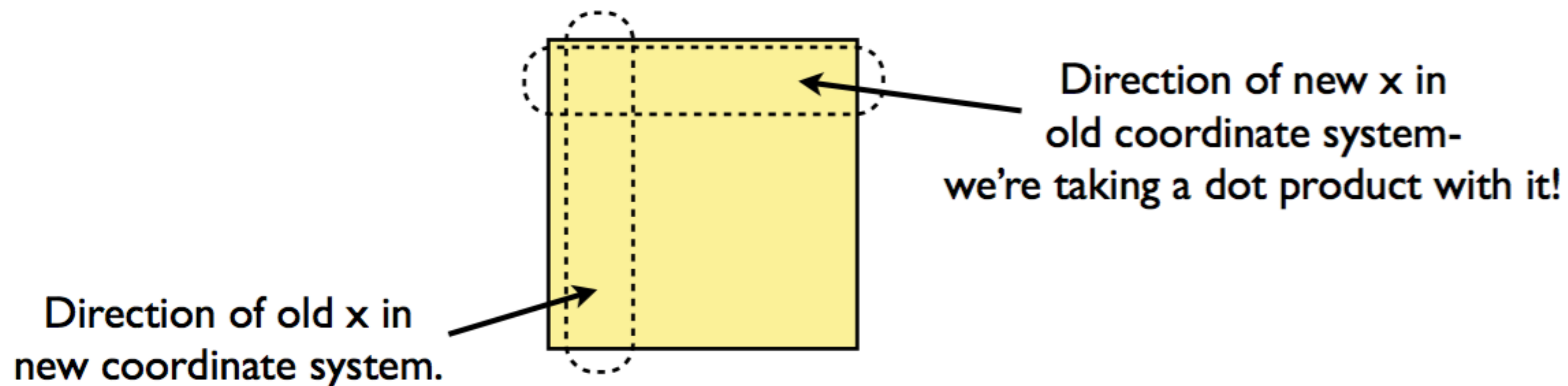
Matrices as projections

- Matrix multiplication projects from one space to another.


$$\text{Cyan Rectangle} = A \text{ Blue Rectangle}$$

Data projected into new
coordinate system

Data in original
coordinate system



Notable Matrices and Operations

- Matrix identity (I) causes no change: $A = I_m A = A I_n$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Diagonal elements $A_{ii} = 1$
- Off-diagonal elements $A_{ij} = 0, i \neq j$
- Matrix inverse (A^{-1}): if $AA^{-1} = A^{-1}A = I$

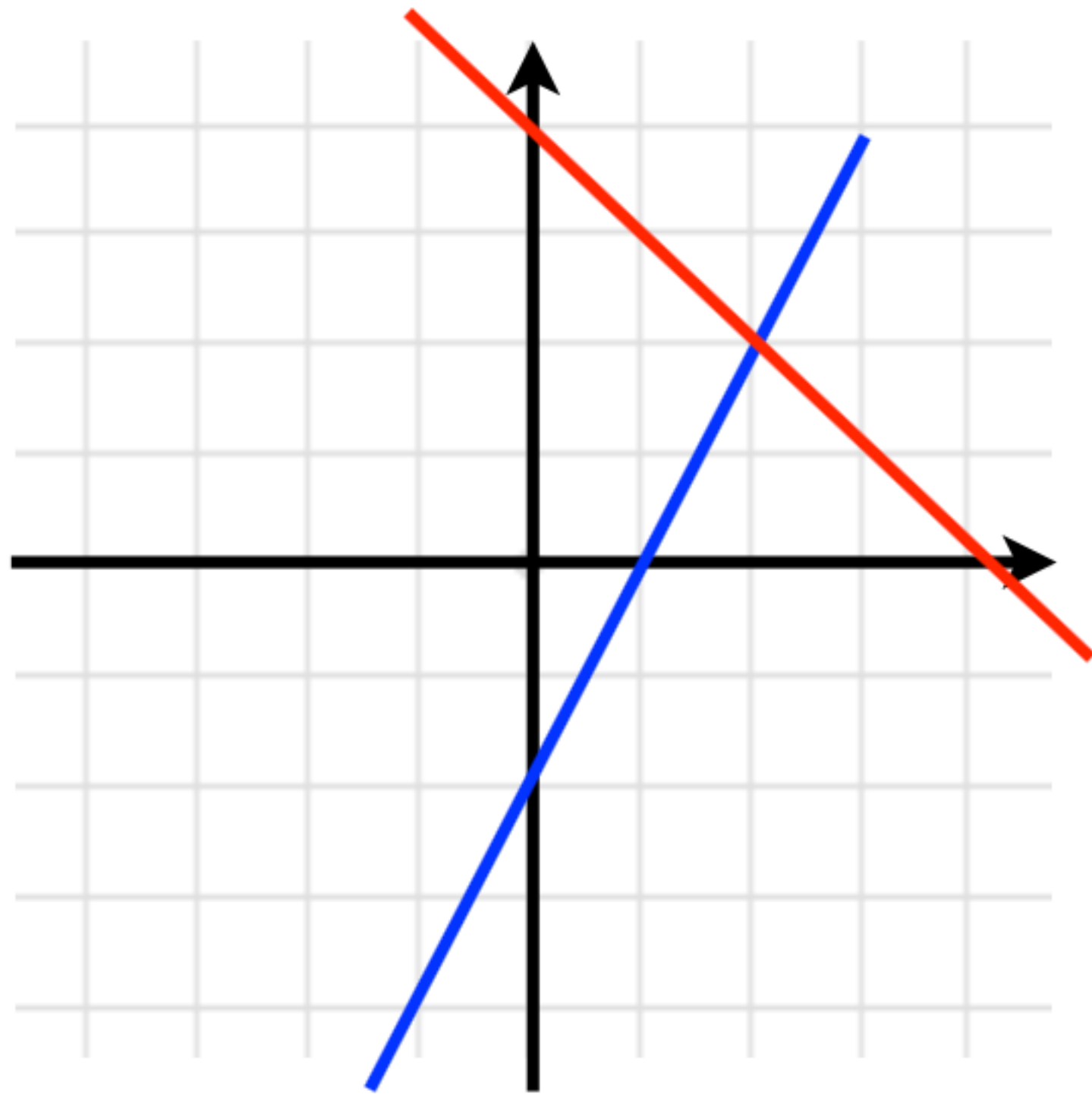
- Distributing matrix inverse: $(AB)^{-1} = B^{-1}A^{-1}$

- Matrix transpose (A^T): a matrix's reflection about its diagonal

- Distributing matrix transpose: $(AB)^T = B^T A^T$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Matrix Geometry: Row Story

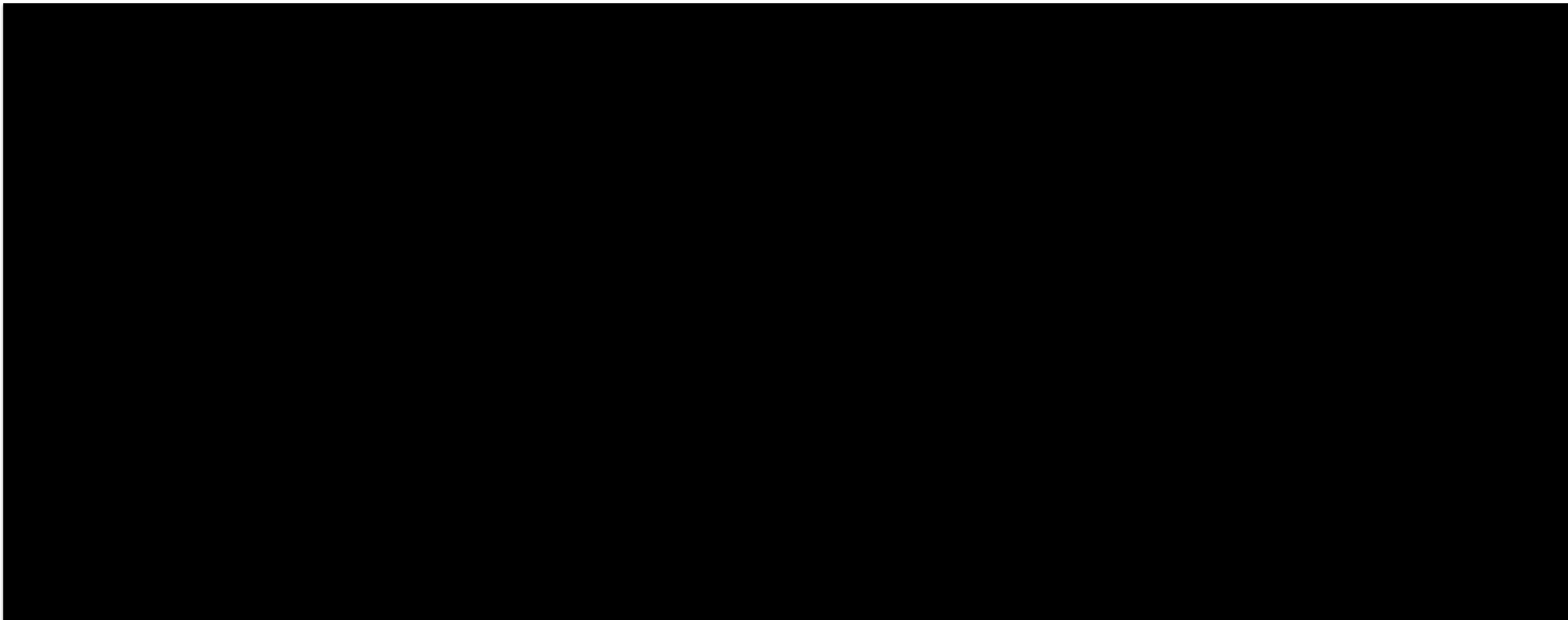


- Each row of a linear system represents a hyperplane. (In 2D, that's also a line!)
- The solution to the system is the intersection of those hyperplanes

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} x = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Solving linear systems

What would be the direct way to solve for $\mathbf{x}$? $A\mathbf{x} = \mathbf{b}$



Solving linear systems

What would be the direct way to solve for $\mathbf{x}$?

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

Invert $\mathbf{A}$ and multiply by $\mathbf{b}$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Matrix rank and inversion

- Let A be a square n by n matrix. A is invertible if full rank and a matrix B exists such that
- Rank of a matrix A is the size of the largest collection of linearly independent columns of A
- A is invertible (nonsingular) if it has full rank
- Gaussian elimination can find matrix inverse
- Singular matrix cannot be inverted this way

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$$

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array} \right]$$

$$[I|B] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array} \right]$$

Solution by Decomposition

- In real applications, inverse not computed to solve linear systems
 - Efficiency, numerical precision, etc.
- Matrix decomposed into product of lower and upper triangular matrices
 - LU decomposition $\mathbf{A} = \mathbf{LU}$
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$
 - Cholesky decomposition $\mathbf{A} = \mathbf{LL}^T$
- Permits finding solution by forward substitution $\mathbf{Ly} = \mathbf{b}$
followed by backward substitution $\mathbf{L}^T \mathbf{x} = \mathbf{y}$

Solving linear systems

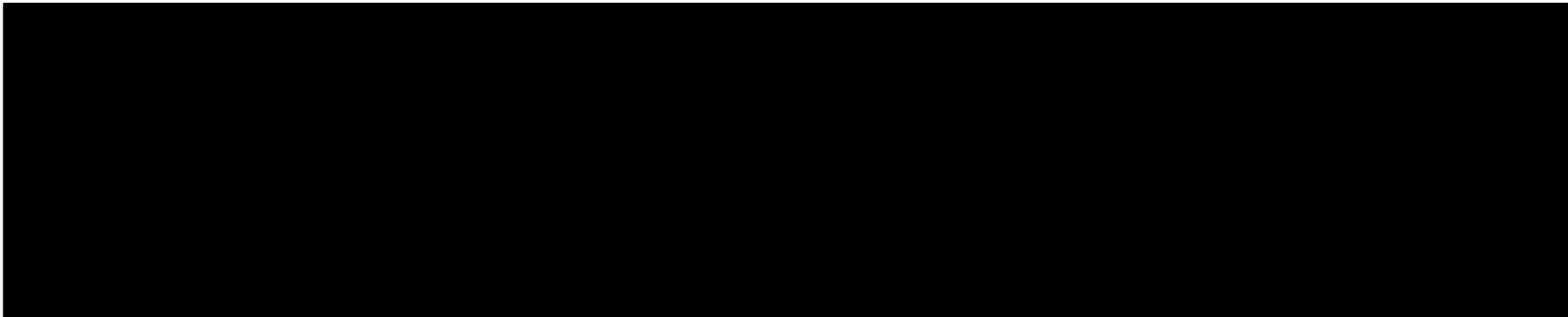
What would be the direct way to solve for $\mathbf{x}$?

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

Invert $\mathbf{A}$ and multiply by $\mathbf{b}$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Can this always be done?



Solving linear systems

What would be the direct way to solve for $\mathbf{x}$?

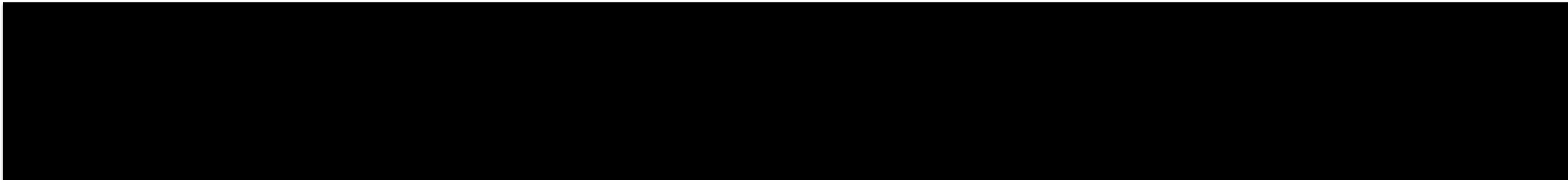
$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

Invert $\mathbf{A}$ and multiply by $\mathbf{b}$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Can this always be done?

No. But, we can approximate. How?



Solving linear systems

What would be the direct way to solve for $\mathbf{x}$?

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

Invert $\mathbf{A}$ and multiply by $\mathbf{b}$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Can this always be done?

No. But, we can approximate. How?

Pseudoinverse least-squares approximation

$$\mathbf{x} = \mathbf{A}_{\text{left}}^+ \mathbf{b}$$

Pseudoinverse

- For matrix A with dimensions $N \times M$ with full rank
- Find solution that minimizes squared error: $\|Ax - b\|_2$
- Left pseudoinverse, for when $N > M$, (i.e., “tall”)

$$A_{\text{left}}^{-1} = (A^T A)^{-1} A^T \quad \text{s.t.} \quad A_{\text{left}}^{-1} A = I_n$$

- Right pseudoinverse, for when $N < M$, (i.e., “wide”)

$$A_{\text{right}}^{-1} = A^T (A A^T)^{-1} \quad \text{s.t.} \quad A A_{\text{right}}^{-1} = I_m$$

Polynomial Regression

- Given n data points as input-output (x_i, y_i) , estimate parameters β of best fitting m -order polynomial:

$$y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \cdots + \beta_m x_i^m \quad (i = 1, 2, \dots, n)$$

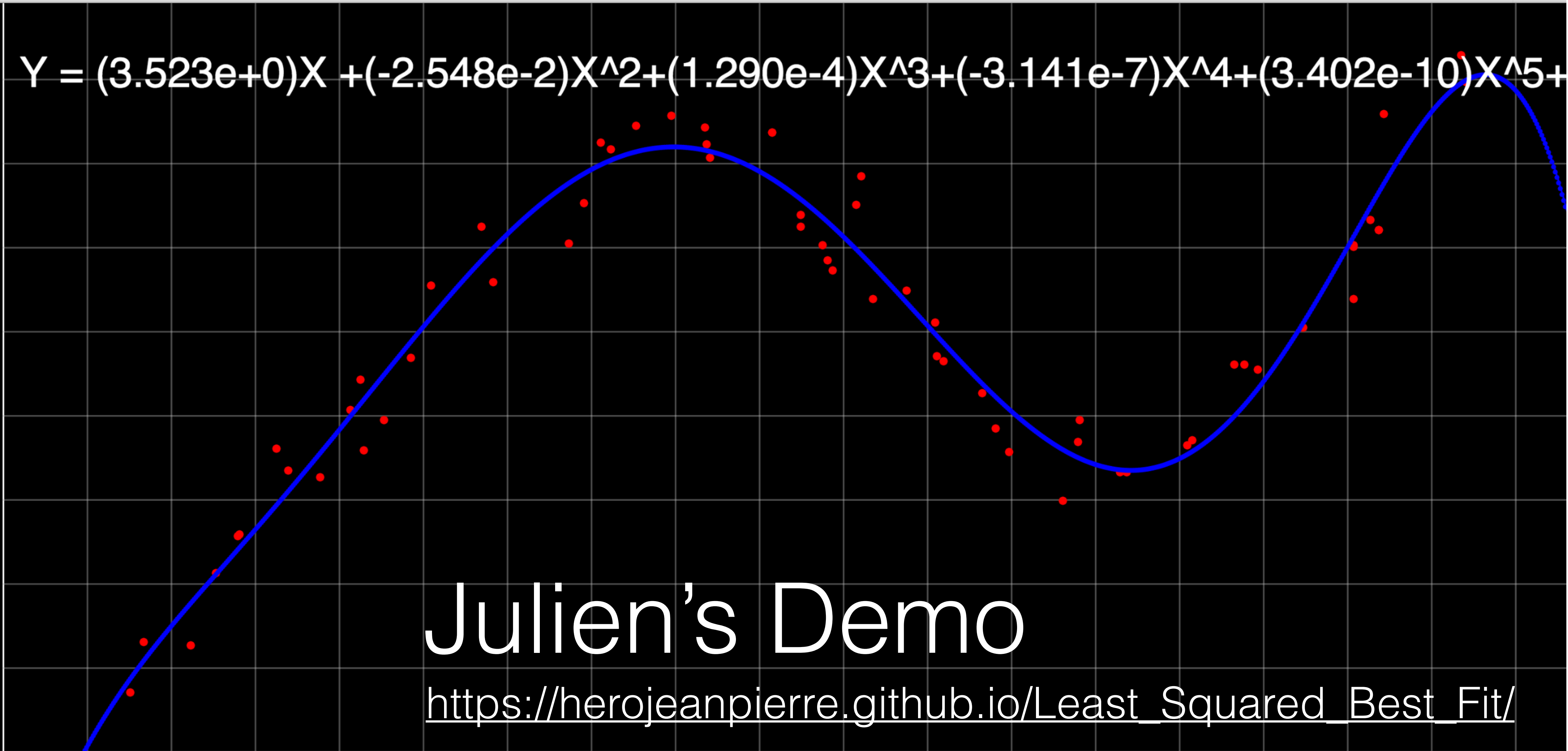
- Model in matrix form:

- each data point forms a row

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^m \\ 1 & x_2 & x_2^2 & \cdots & x_2^m \\ 1 & x_3 & x_3^2 & \cdots & x_3^m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^m \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix}$$

- Solve for least squares best fit: $\hat{\vec{\beta}} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \vec{y},$

$$Y = (3.523e+0)X + (-2.548e-2)X^2 + (1.290e-4)X^3 + (-3.141e-7)X^4 + (3.402e-10)X^5 +$$



Julien's Demo

https://herojeanpierre.github.io/Least_Squared_Best_Fit/

Clear

- Order

+ Order

Order: 6

Determinant

- The determinant $\det(A)$ (or $|A|$) of a matrix A is a scalar value
- Scaling factor of the linear transformation described by matrix A

- For 2-by-2 matrix: $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

- For 3-by-3 matrix: $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

- For N -by- N matrix: $\det(A) = \sum_{\sigma \in S_n} \left(\text{sgn}(\sigma) \prod_{i=1}^n a_{i,\sigma_i} \right)$

Eigenvalues and Eigenvectors

- Vector $\mathbf{v}$ and scalar λ are an eigenvector and eigenvalue for matrix A if $(A - \lambda I)\mathbf{v} = 0$
- $(A - \lambda I) = 0$ has a non-zero solution $\mathbf{v}$ if and only if $\det(A - \lambda I) = 0$
- $\det(A - \lambda I)$ produces characteristic polynomial, whose roots are the eigenvalues

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

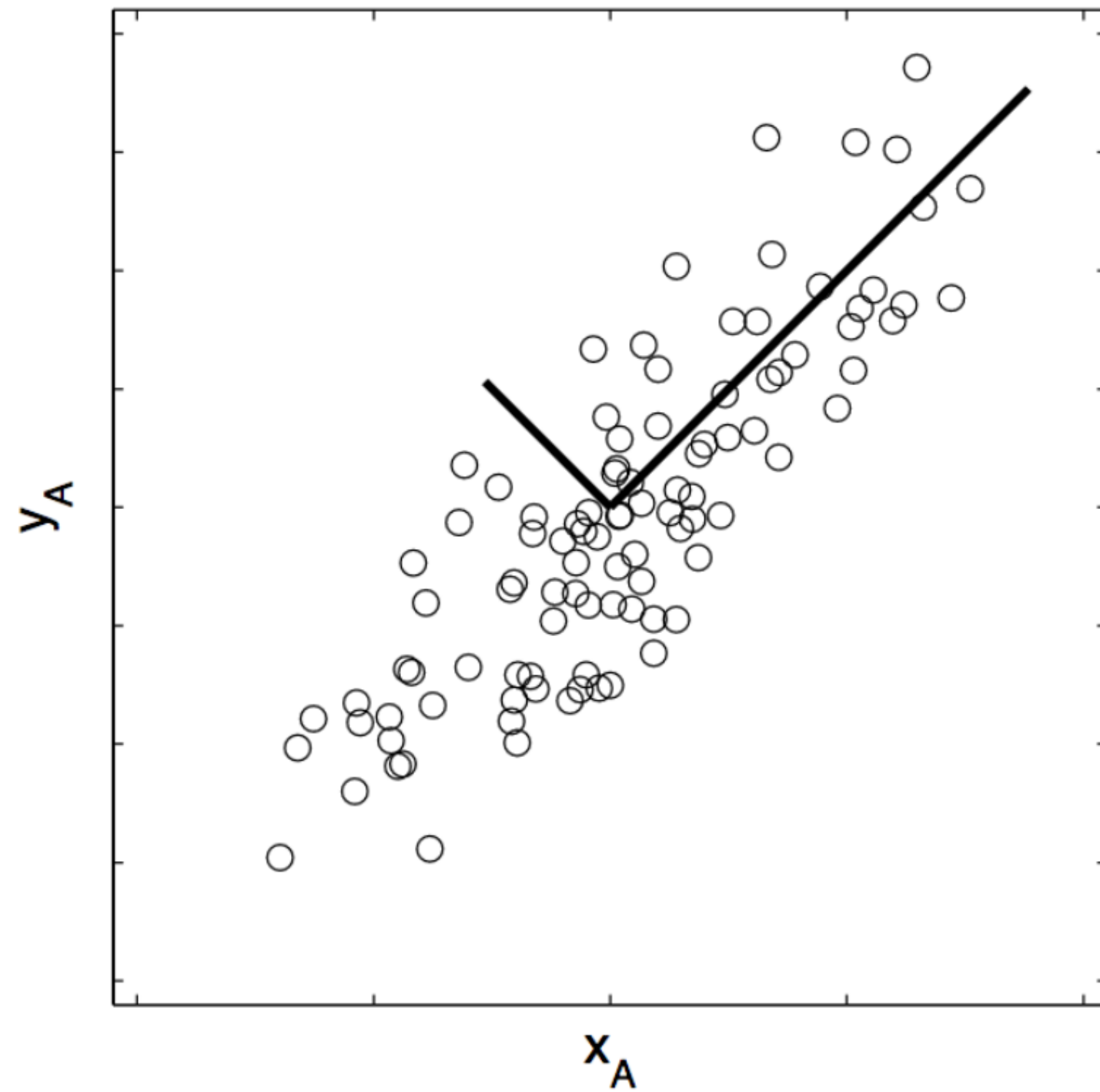
$$|M - \lambda I| = \begin{vmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix} = 3 - 4\lambda + \lambda^2$$

$$\mathbf{v}_{\lambda=1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

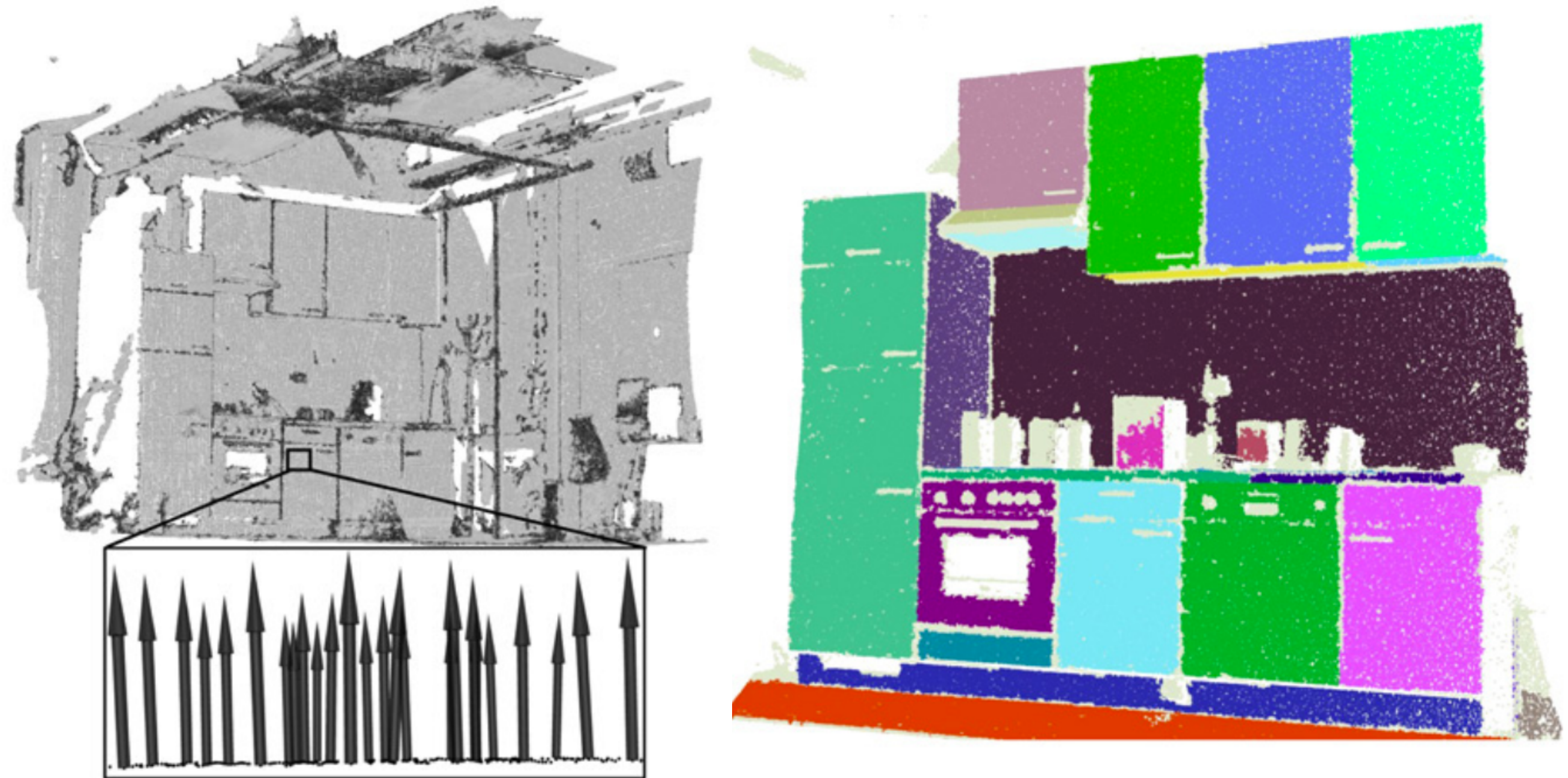
$$\mathbf{v}_{\lambda=3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Principal Components

- The eigenvectors of the D -by- D covariance matrix XX^T of N data points observed in a D dimensional space, scaled by the eigenvalues



Shlens 2005



PCA-based flat surface estimator - Rusu et al. 2008

Next Lecture

