Dynamics and Numerical Integration

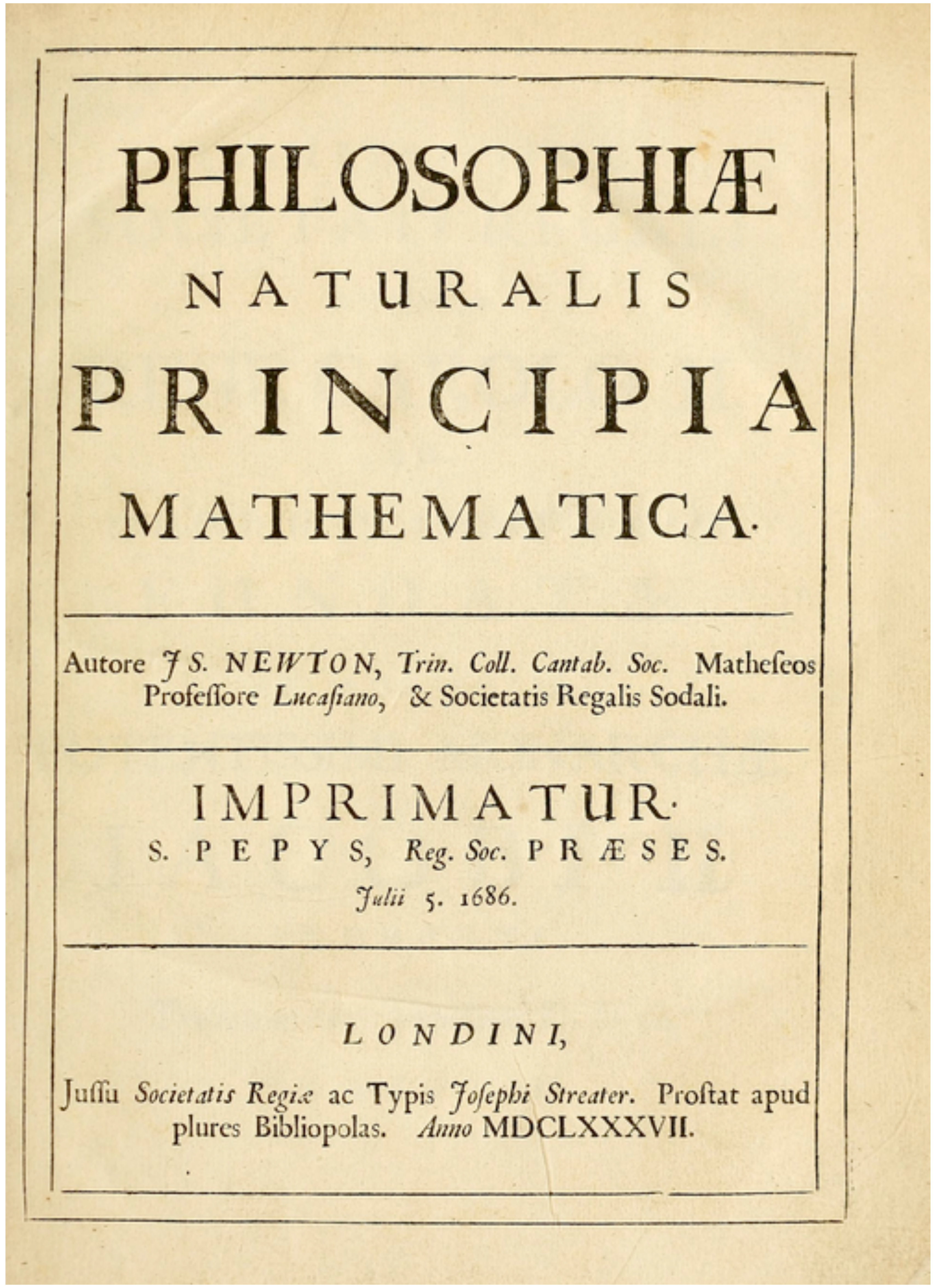
autorob.org

EECS 367
Intro. to Autonomous Robotics

ROB 320
Robot Operating Systems

Winter 2022

1687


PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.

Autore J S. NEWTON, *Trin. Coll. Cantab. Soc.* Matheſeos Profeſſore *Lucaſiano*, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, *Reg. Soc.* PRÆSES.
Julii 5. 1686.

LONDINI,

Juſſu *Societatis Regiæ* ac Typis *Joſephi Streater*. Proſtat apud plures Bibliopolas. *Anno* MDCLXXXVII.

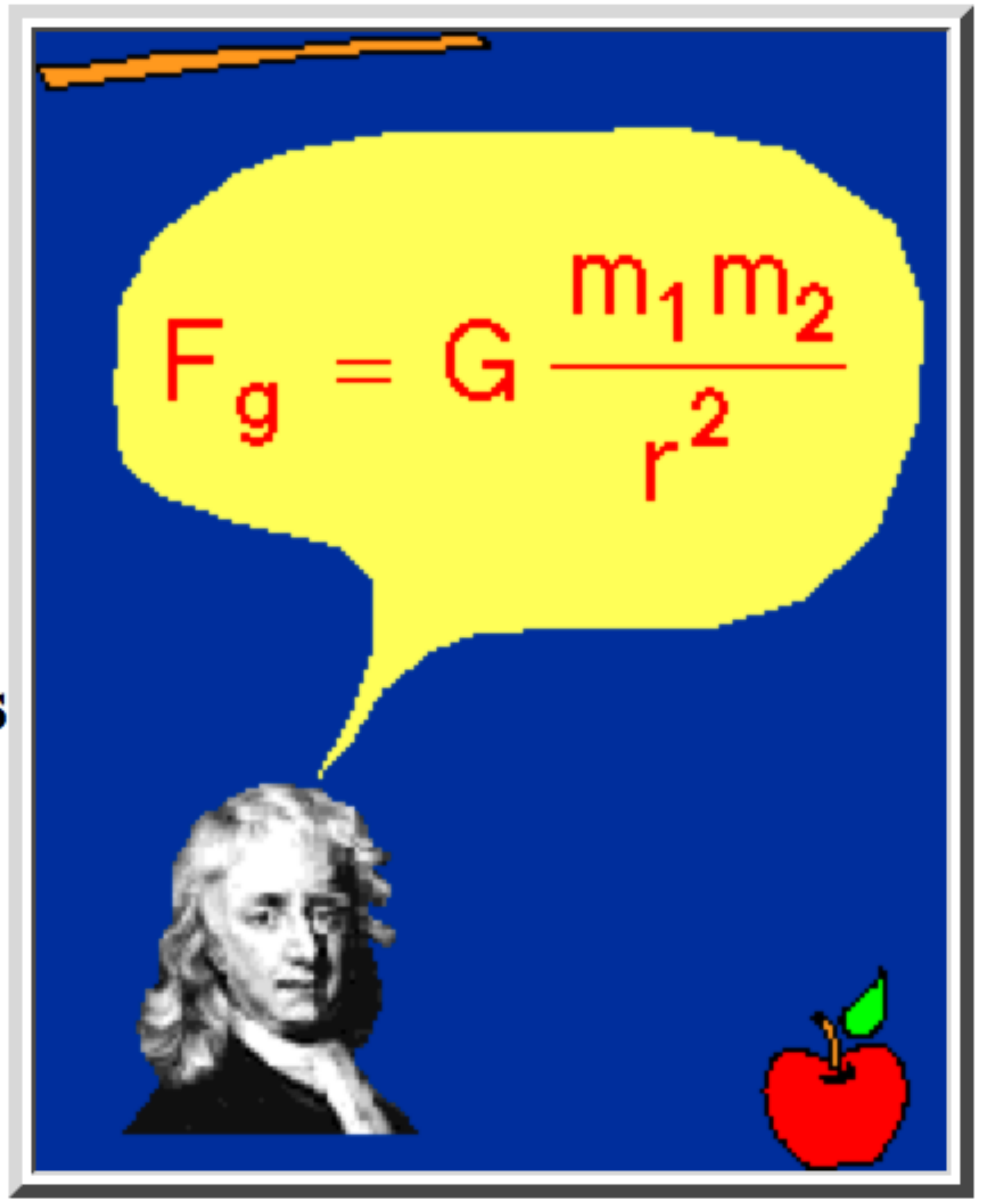
2012

http://drawception.com/viewgame/azF19y6c8s/edible-science/

There is a popular story that Newton was sitting under an apple tree, an apple fell on his head, and he suddenly thought of the Universal Law of Gravitation. As in all such legends, this is almost certainly not true in its details, but the story contains elements of what actually happened.

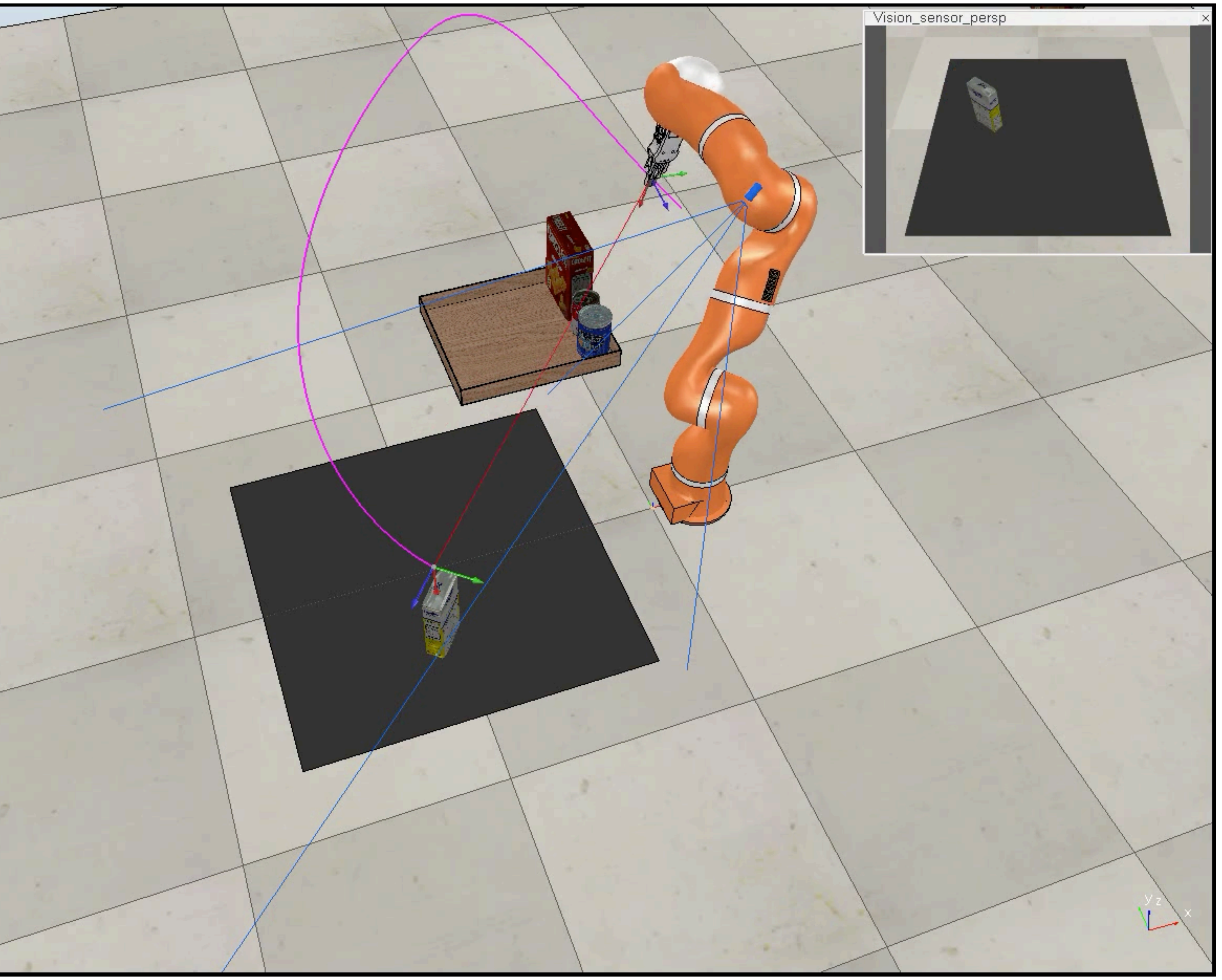
What Really Happened with the Apple?

Probably the more correct version of the story is that Newton, upon observing an apple fall from a tree, began to think along the following lines: The apple is accelerated, since its velocity changes from zero as it is hanging on the tree and moves toward the ground. Thus, by Newton's 2nd Law there must be a force that acts on the apple to cause this acceleration. Let's call this force "gravity", and the associated acceleration the "accleration due to gravity". Then imagine the apple tree is twice as high. Again, we expect the apple to be accelerated toward the ground, so this suggests that this force that we call gravity reaches to the top of the tallest apple tree.



http://csep10.phys.utk.edu/astr161/lect/history/newtongrav.html

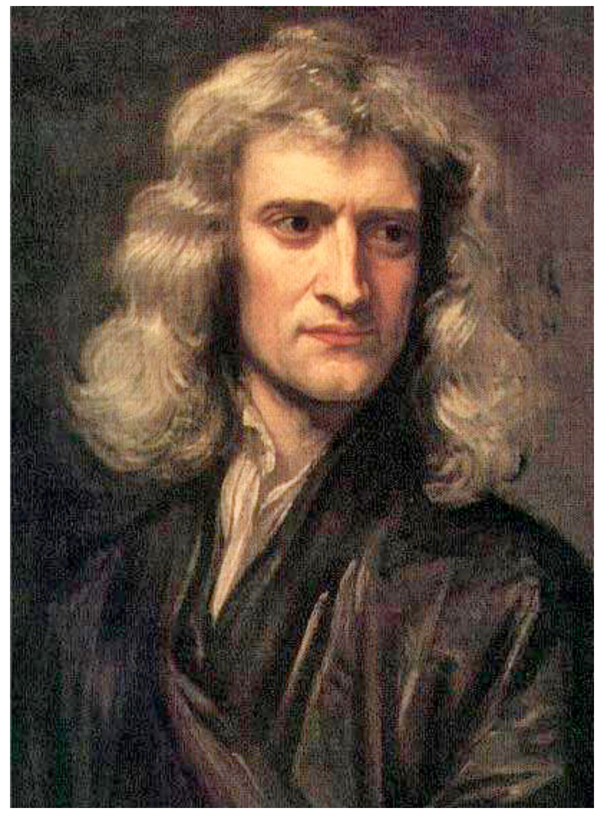
Dynamical Simulation



KUKA Lightweight Arm
in V-REP simulator

Rigid Body Equations of Motion enable simulation of robots on computers

$$\mathbf{F} = m\mathbf{a}_{\mathrm{cm}}$$



Issac Newton
(1643-1727)

$$\mathbf{M} = \mathbf{r}_{\mathrm{cm}} \times \mathbf{a}_{\mathrm{cm}} m + I\boldsymbol{\alpha}$$

Leonhard Euler
(1707-1783)

Users

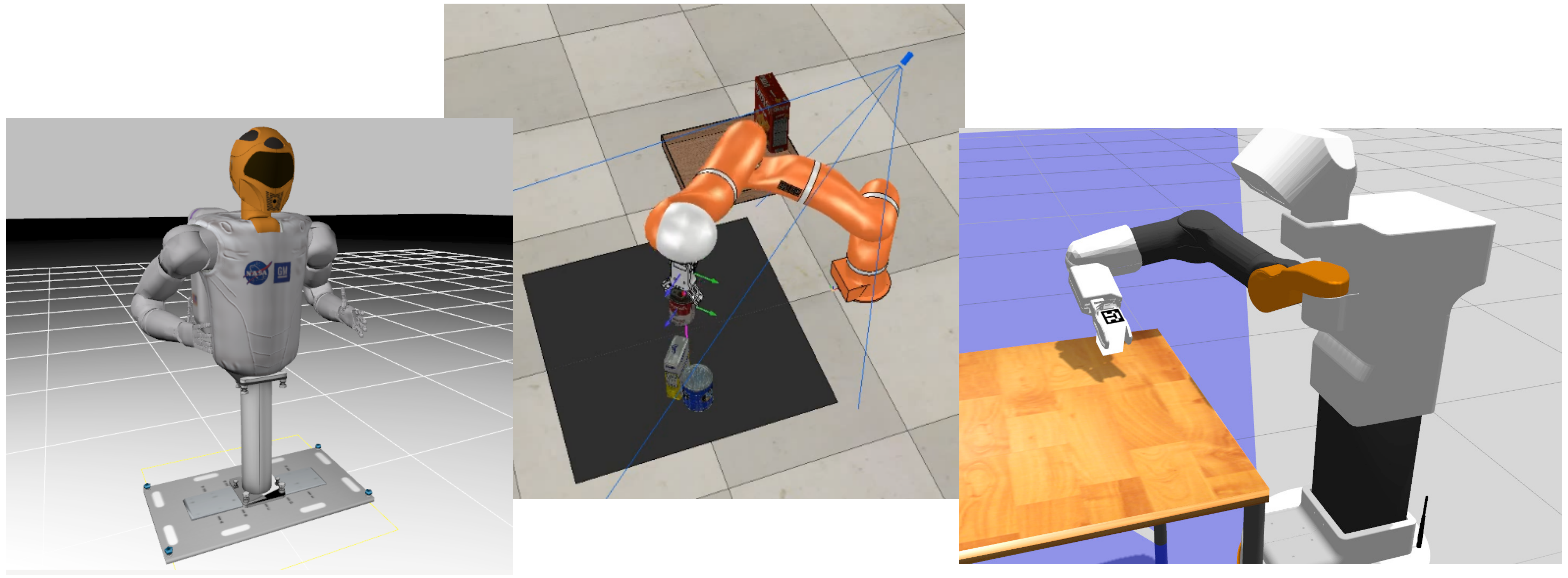
Robot Applications

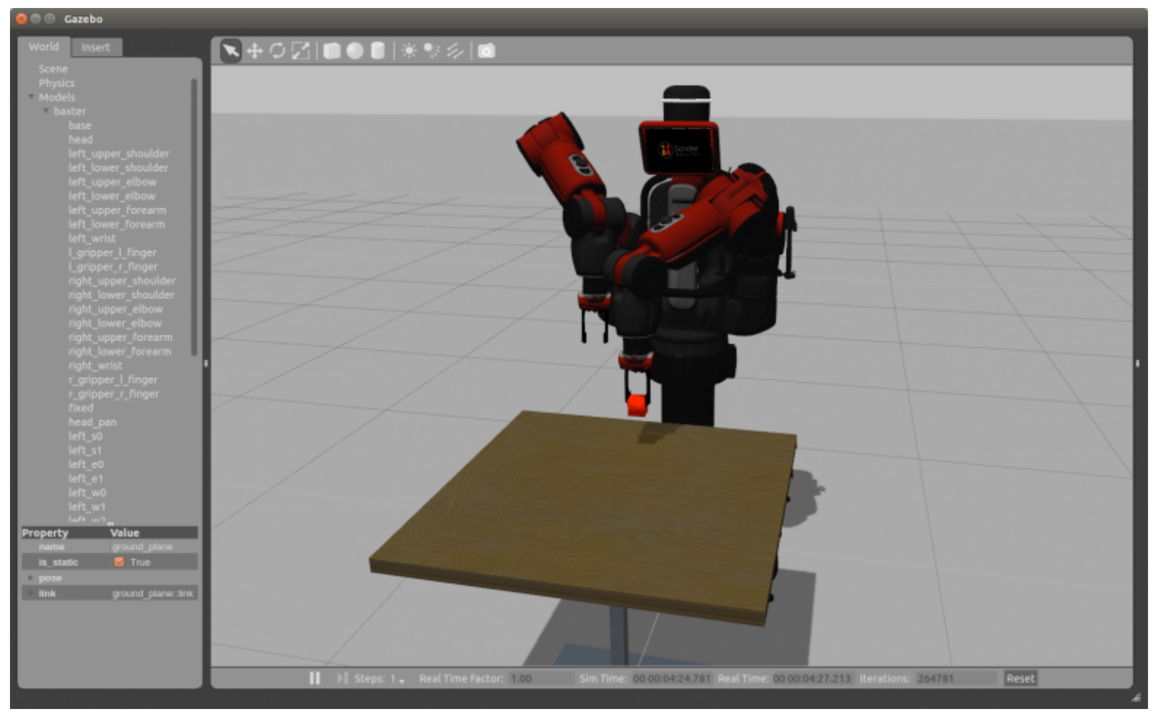
Robot Operating System

Operating System

Dynamical Simulation

$$\mathbf{F} = m\mathbf{a}_{\mathrm{cm}} \qquad \mathbf{M} = \mathbf{r}_{\mathrm{cm}} \times \mathbf{a}_{\mathrm{cm}} m + I\boldsymbol{\alpha}$$





Users

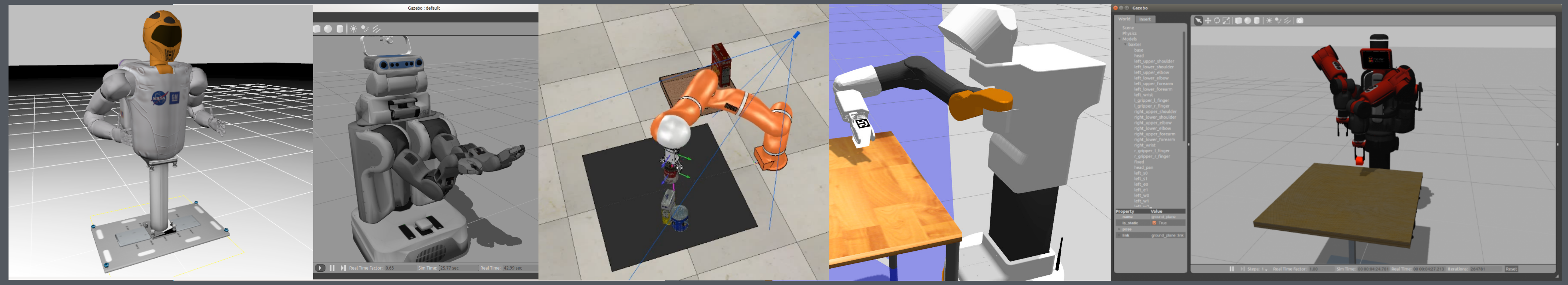
Robot Applications

Robot Operating System

Operating System

Dynamical Simulation

Simulated Hardware



Robot Operating System

Covered at breadth in AutoRob

Classical mechanics

From Wikipedia, the free encyclopedia

In physics, **classical mechanics** and quantum mechanics are the two major sub-fields of mechanics. Classical mechanics is concerned with the set of physical laws describing the motion of bodies under the action of a system of forces. The study of the motion of bodies is an ancient one, making classical mechanics one of the oldest and largest subjects in science, engineering and technology. It is also widely known as **Newtonian mechanics**.

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Classical mechanics

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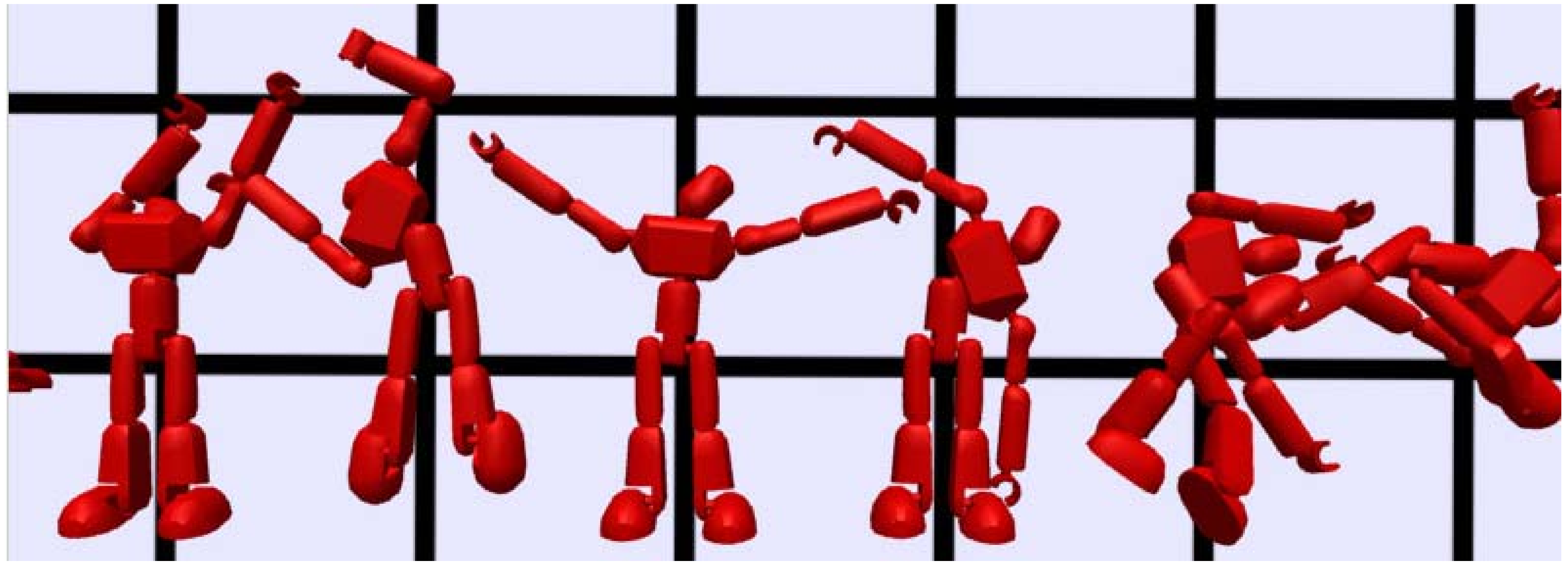
Classical mechanics

physical laws describing the motion of bodies under the action of a system of forces

Kinematics

kin•e•mat•ics | ˌkinəˈmatiks |
plural noun [usu. treated as sing.]
the branch of mechanics concerned with the motion of objects without reference to the forces that cause the motion. Compare with



Dynamics

dy•nam•ics | dīˈnamiks |
plural noun
1 [treated as sing.] the branch of mechanics concerned with the motion of bodies under the action of forces. Compare with STATICS.
• [usu. with modifier] the branch of any science in which forces or changes are considered: *chemical dynamics*.

$F=ma$

Classical mechanics

physical laws describing the motion of bodies under the action of a system of forces

Kinematics

“the geometry of motion”

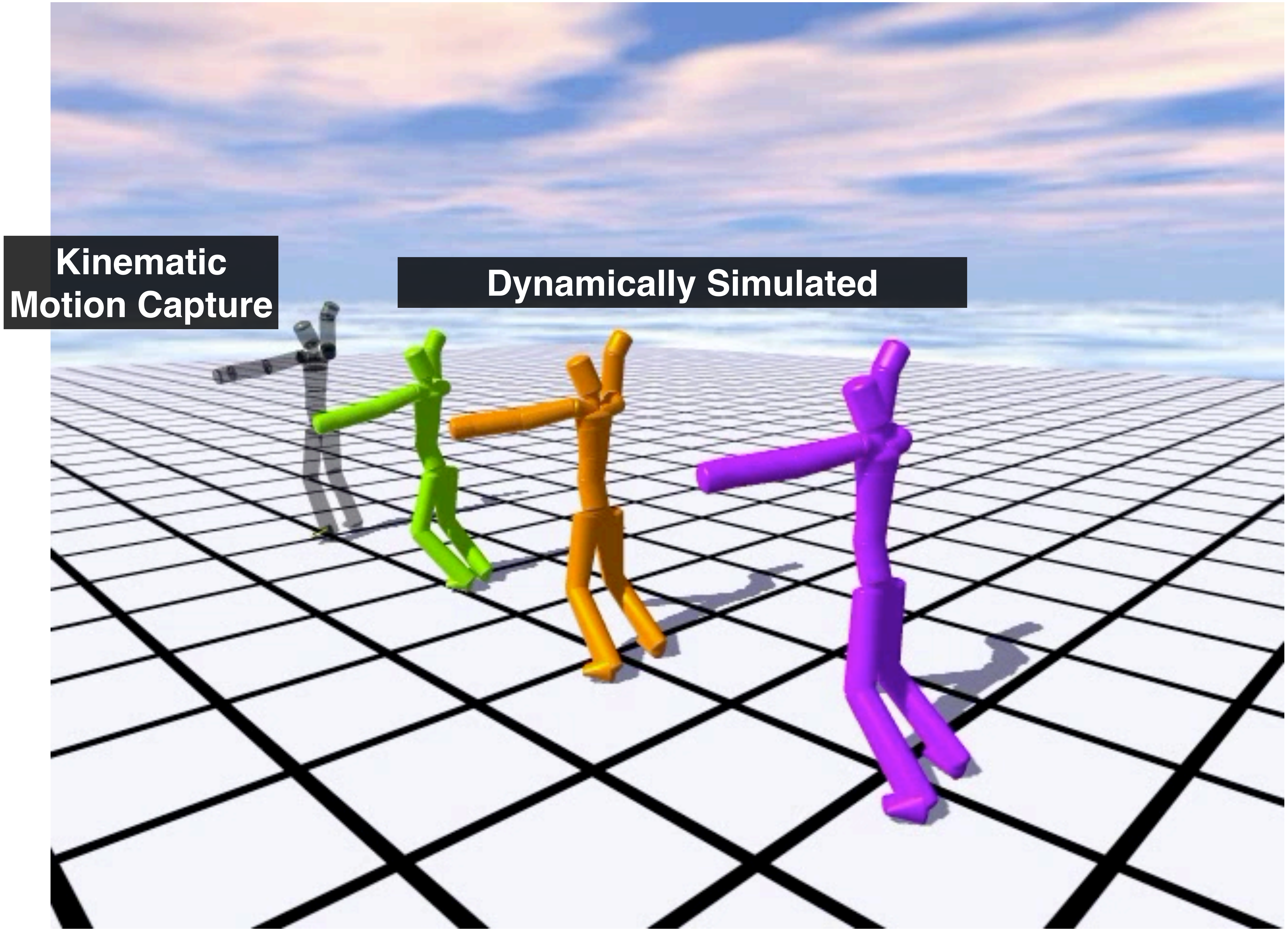
expresses range of possible motion
(or states of the robot)

Dynamics

“physical motion over time”

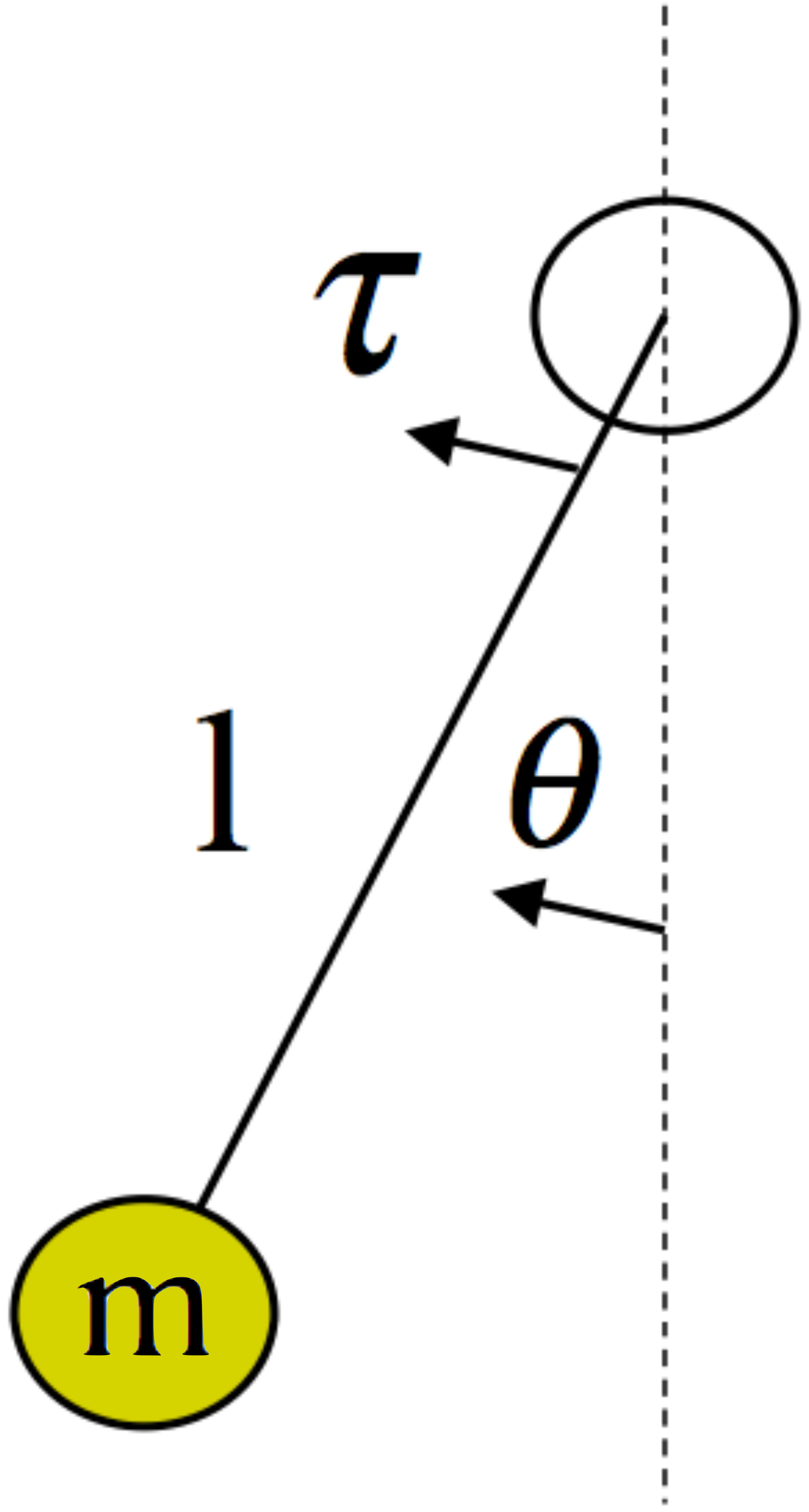
expresses motion over time with
respect to Newton’s laws
(evolution of state over time)

Dynamo - Wrotek, McGuire, Jenkins 2006 - https://youtu.be/5nuhY6kJdv4

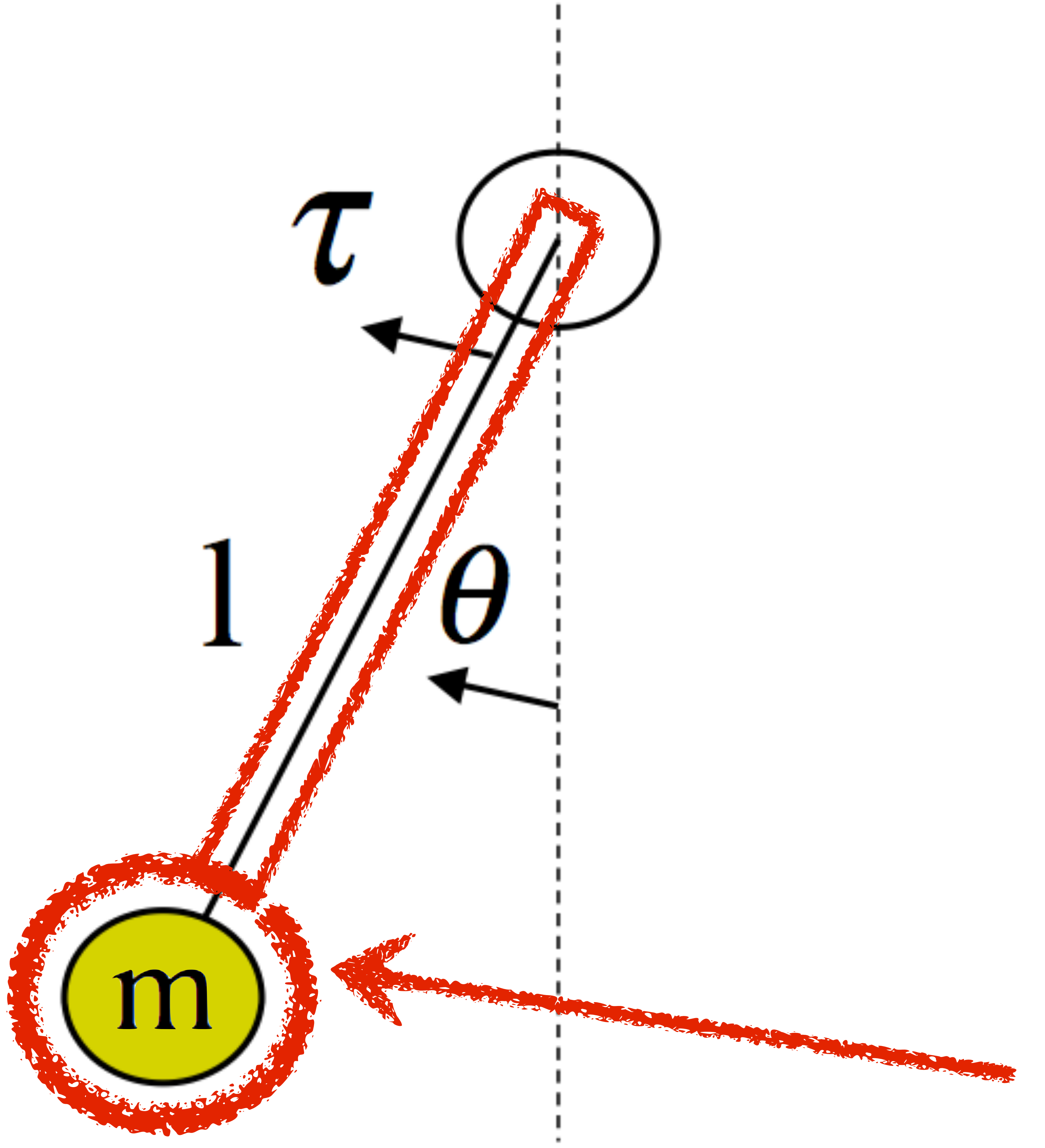


Let’s start with a simple example

Example: Motorized Pendulum

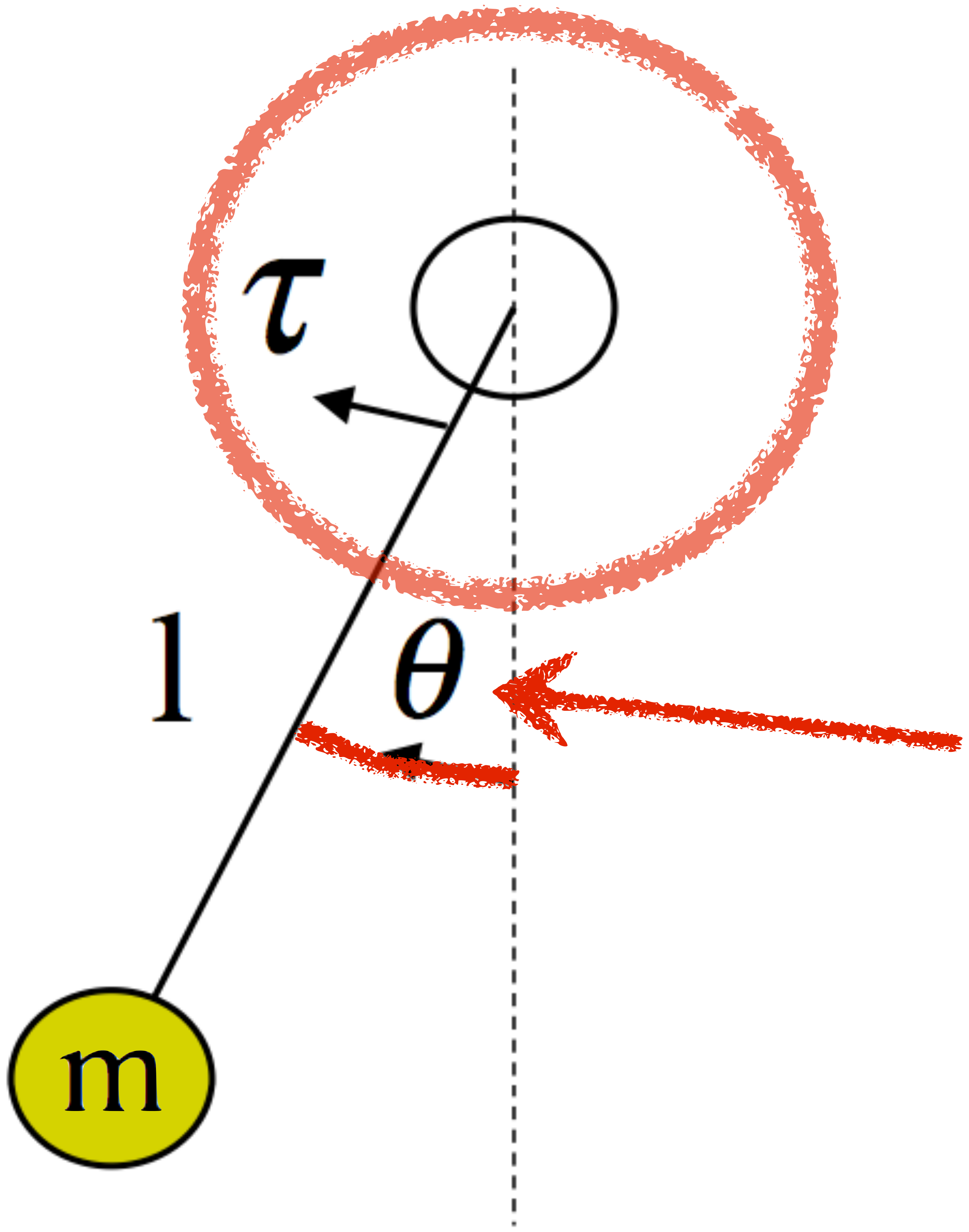


Example: Motorized Pendulum



Pendulum of length *l* with point mass *m*

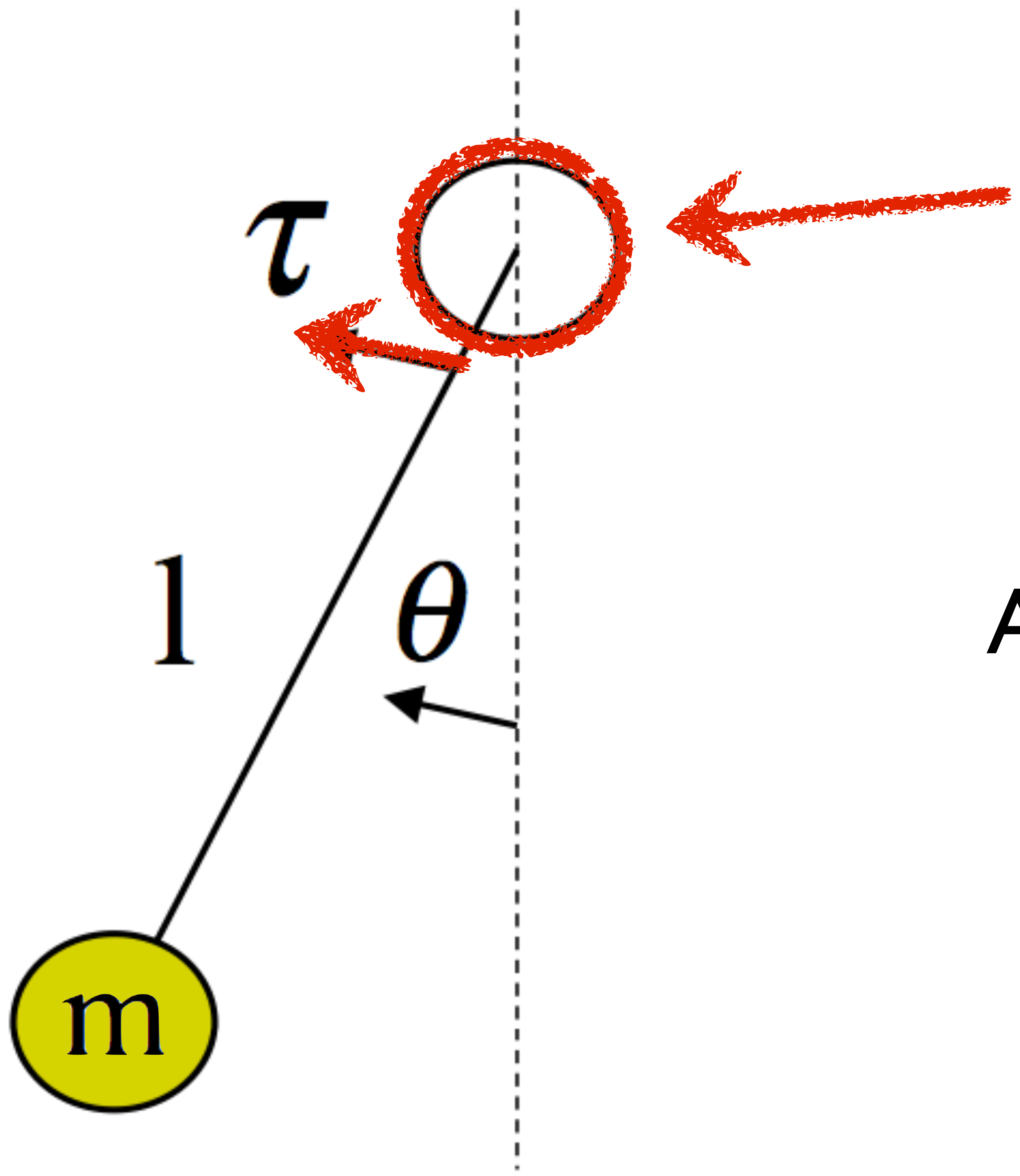
Example: Motorized Pendulum



Angle expresses pendulum range of motion

Pendulum of length *l* with point mass *m*

Example: Motorized Pendulum



Motor produces torque (angular force)

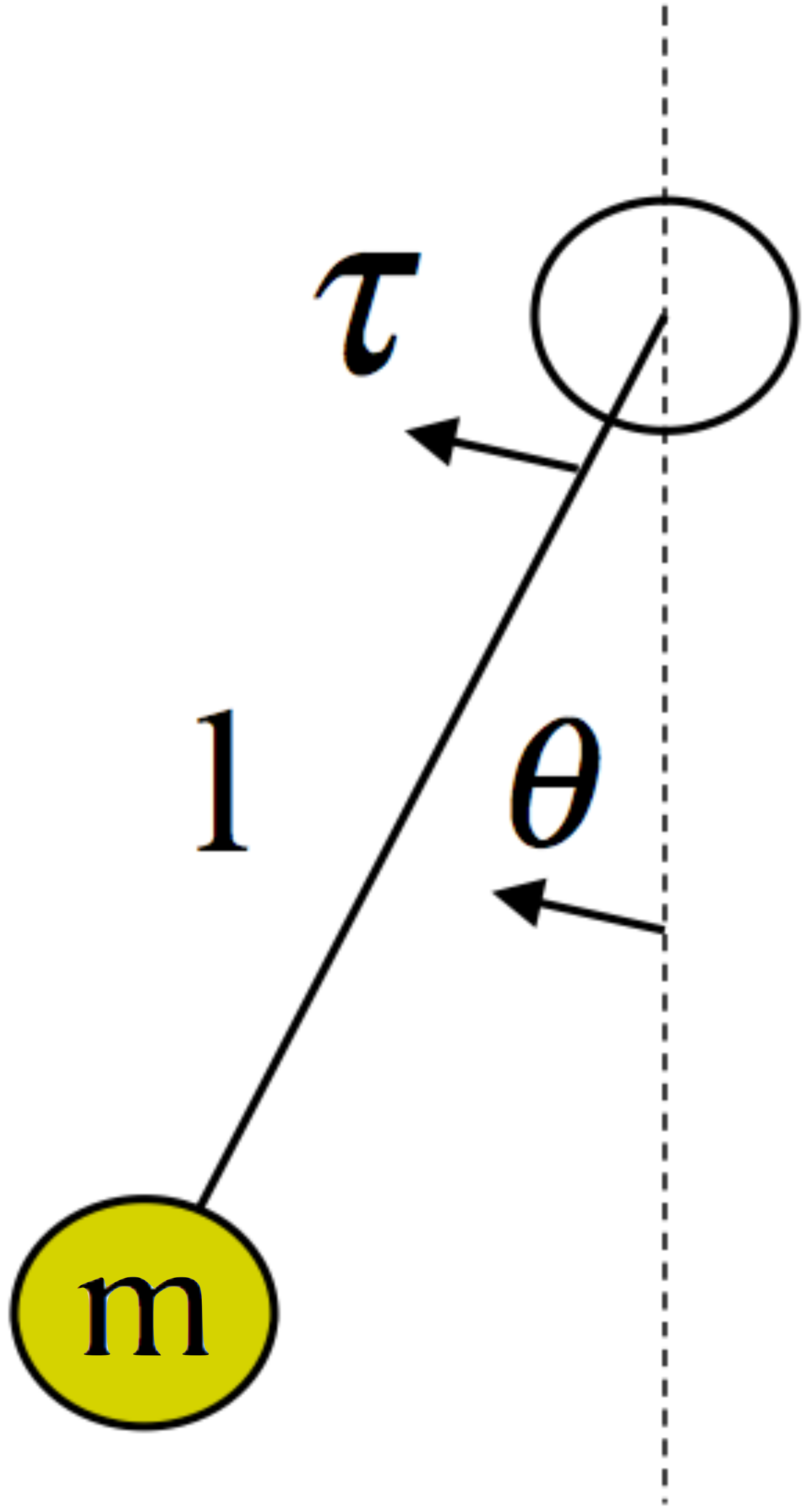
Angle expresses pendulum range of motion

Pendulum of length *l* with point mass *m*

Example: Motorized Pendulum

Equation of motion
(with rotational inertia I)

$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$



Motor produces torque
(angular force)

Angle expresses pendulum
range of motion

Pendulum of length *l* with
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Dynamics

Equation of motion
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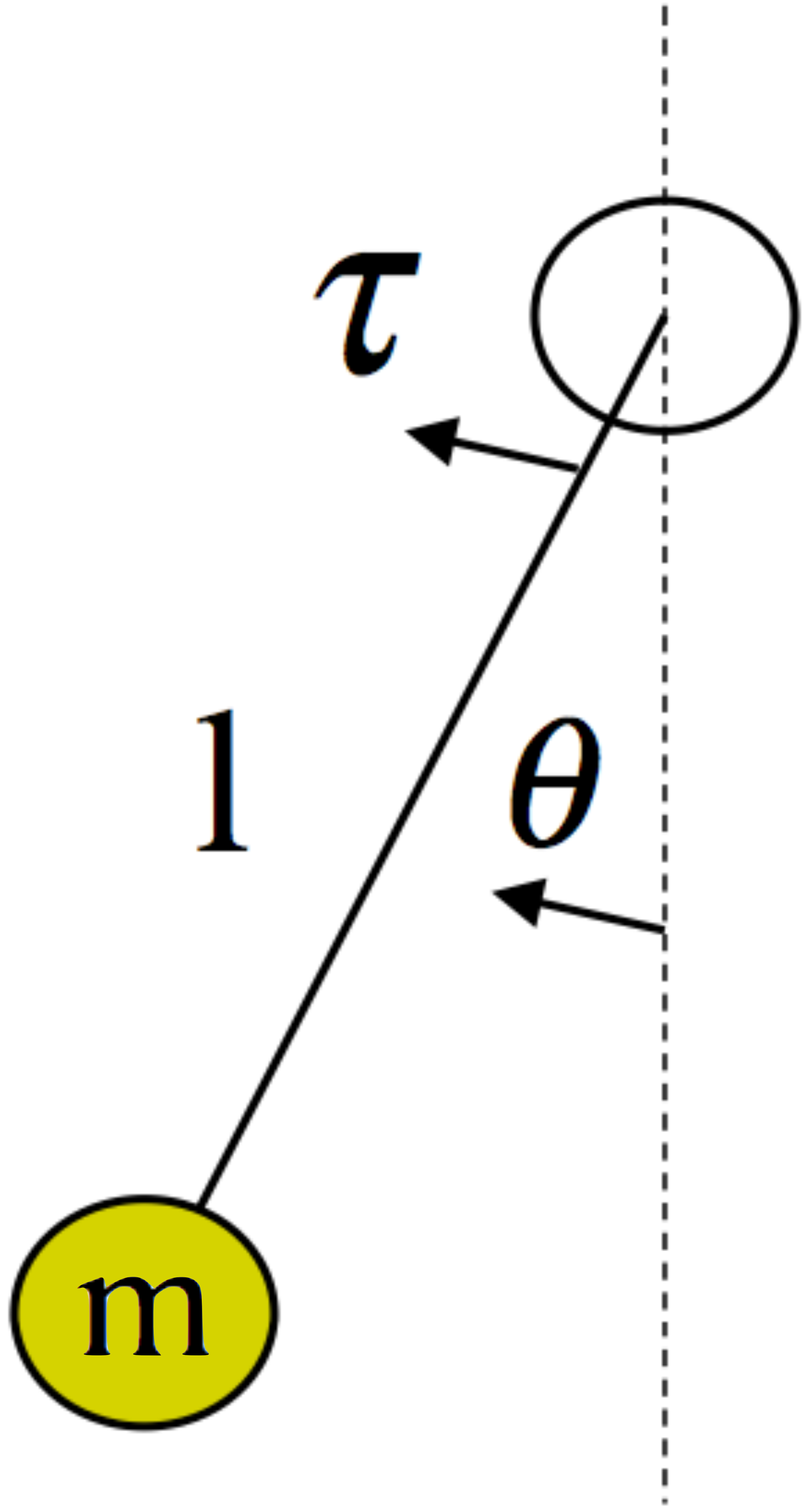


Controls

Motor produces torque
(angular force)

State (or Configuration)

Angle expresses pendulum
range of motion

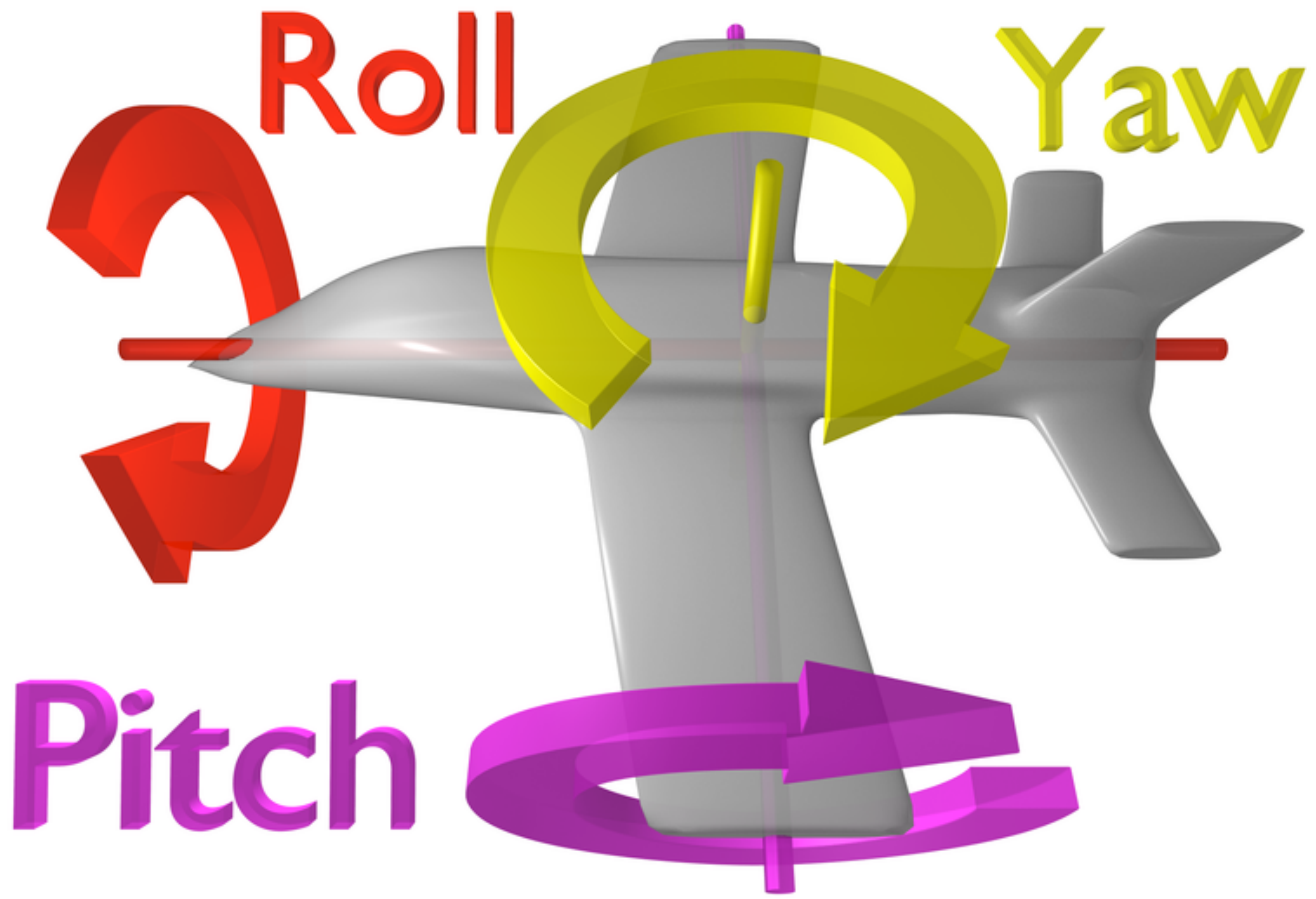
System

Pendulum of length *l* with
point mass *m*

Defining State

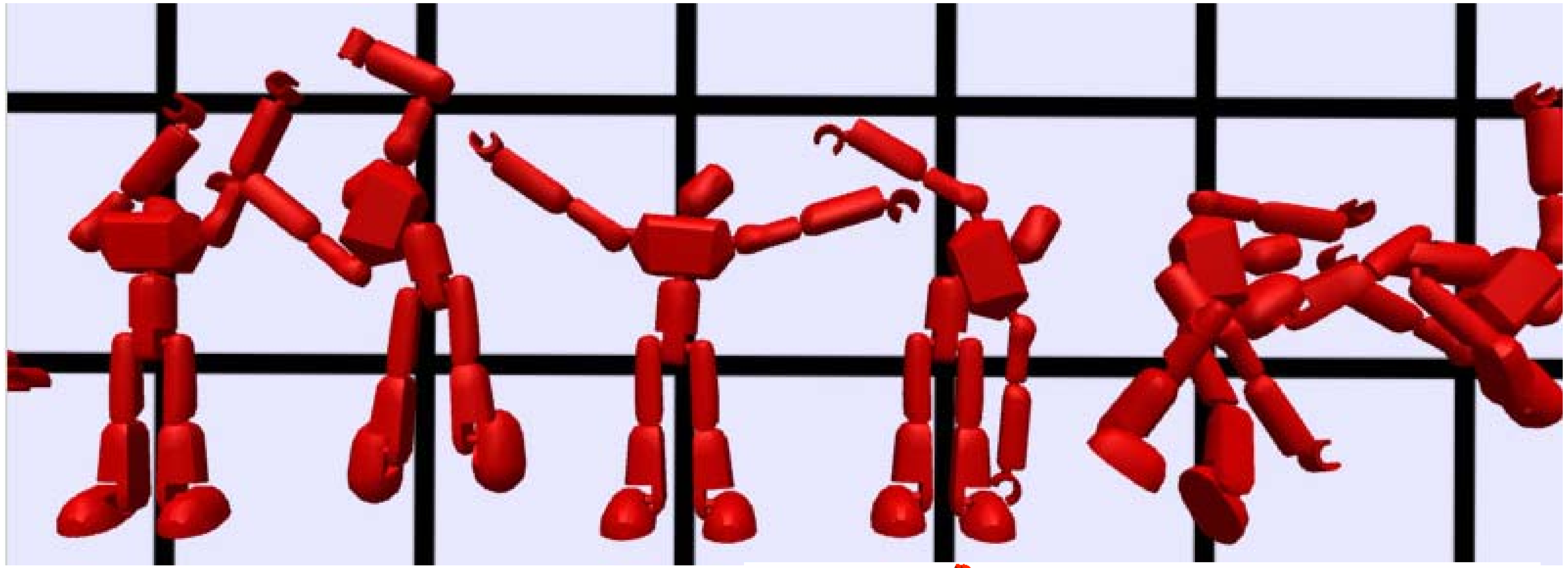
- State comprised of <u>degrees-of-freedom (DOFs)</u>
- DOFs describe translational and rotational axes for the motion about robot joints

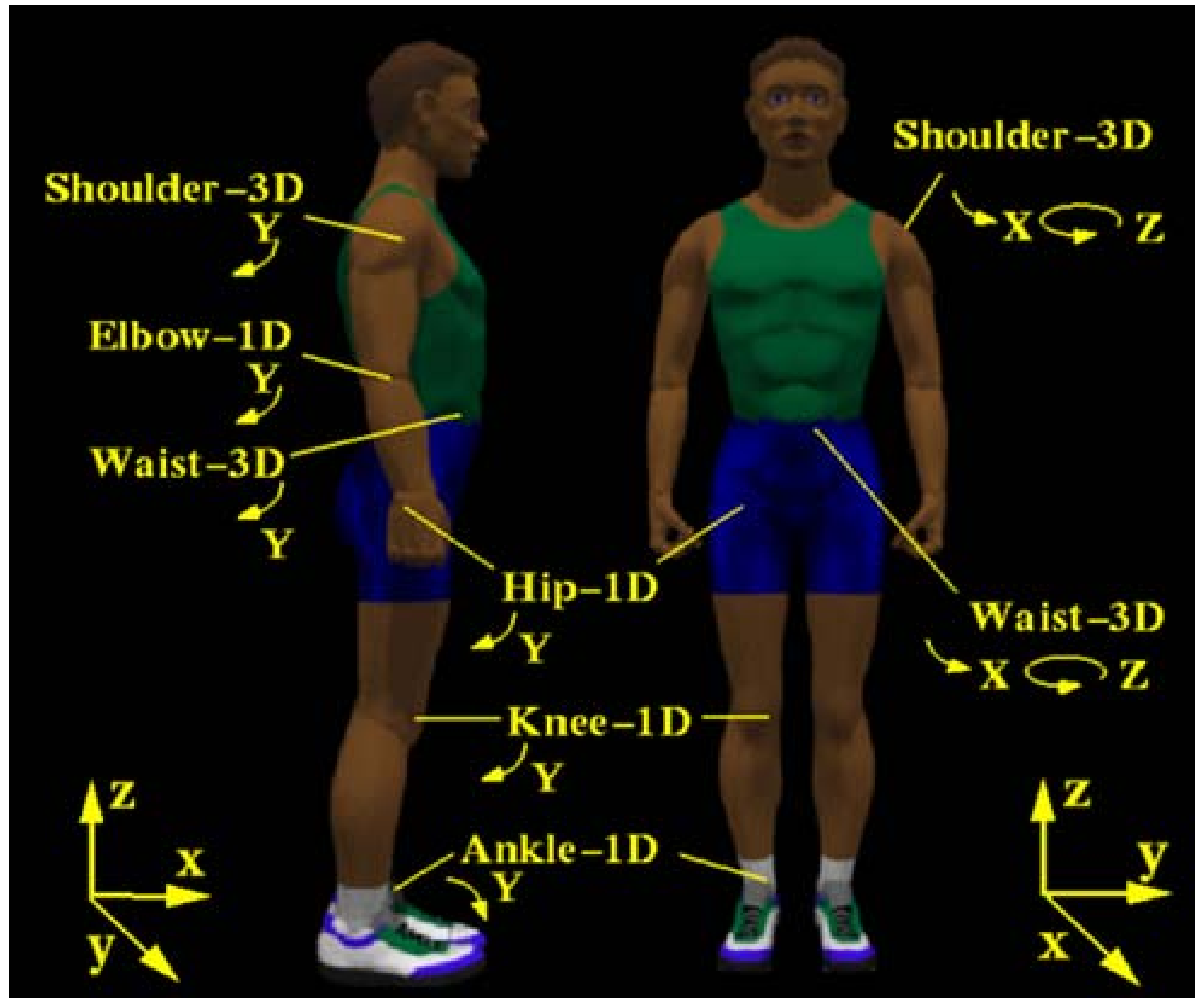
- How many DOFs in the example pendulum?
- Airplane DOFs?

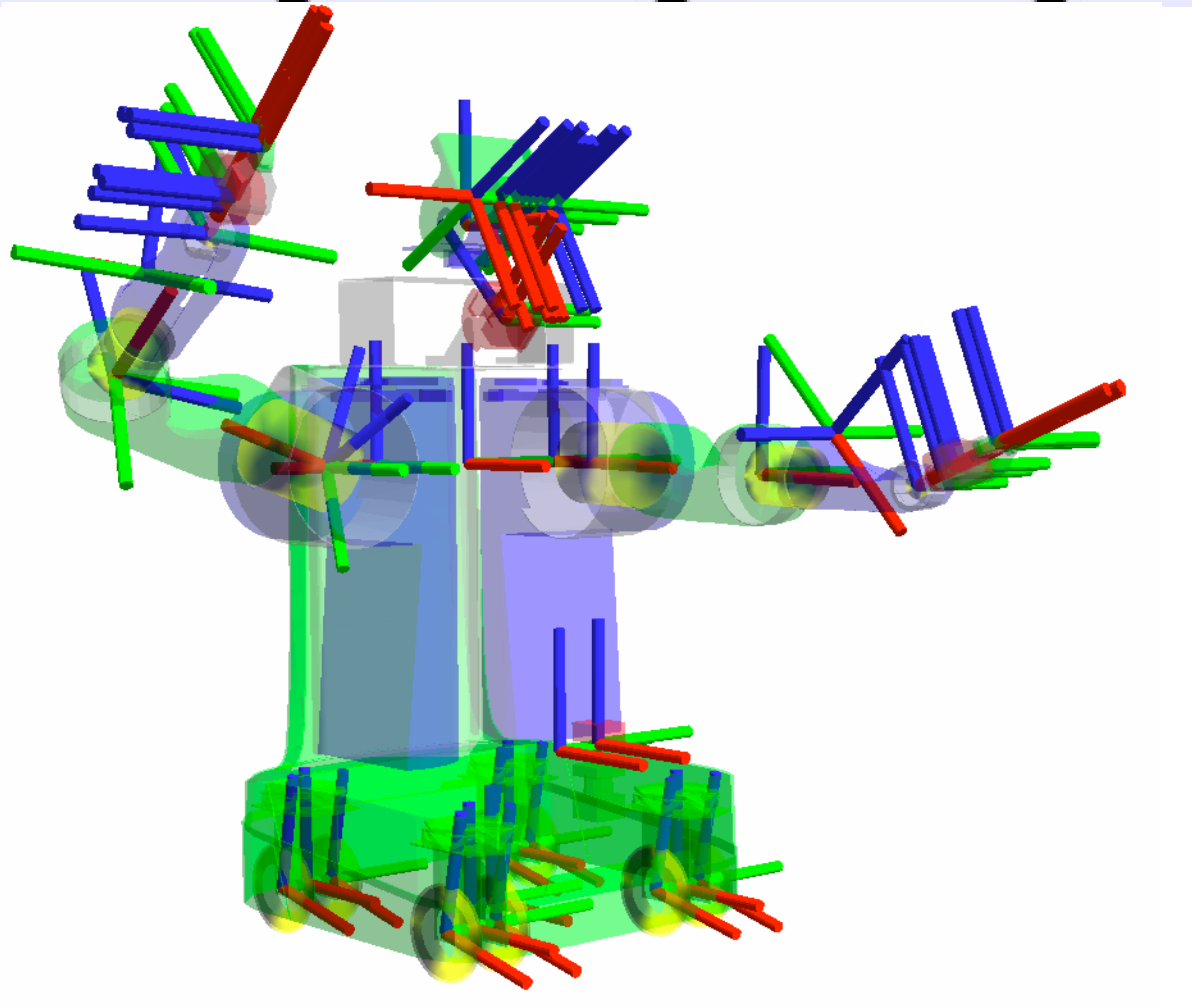


Defining State

- State comprised of degrees-of-freedom (DOFs)
- DOFs describe translational and rotational axes for the motion about robot joints
- Humanoid DOFs?
  - joint angles
  - global positioning





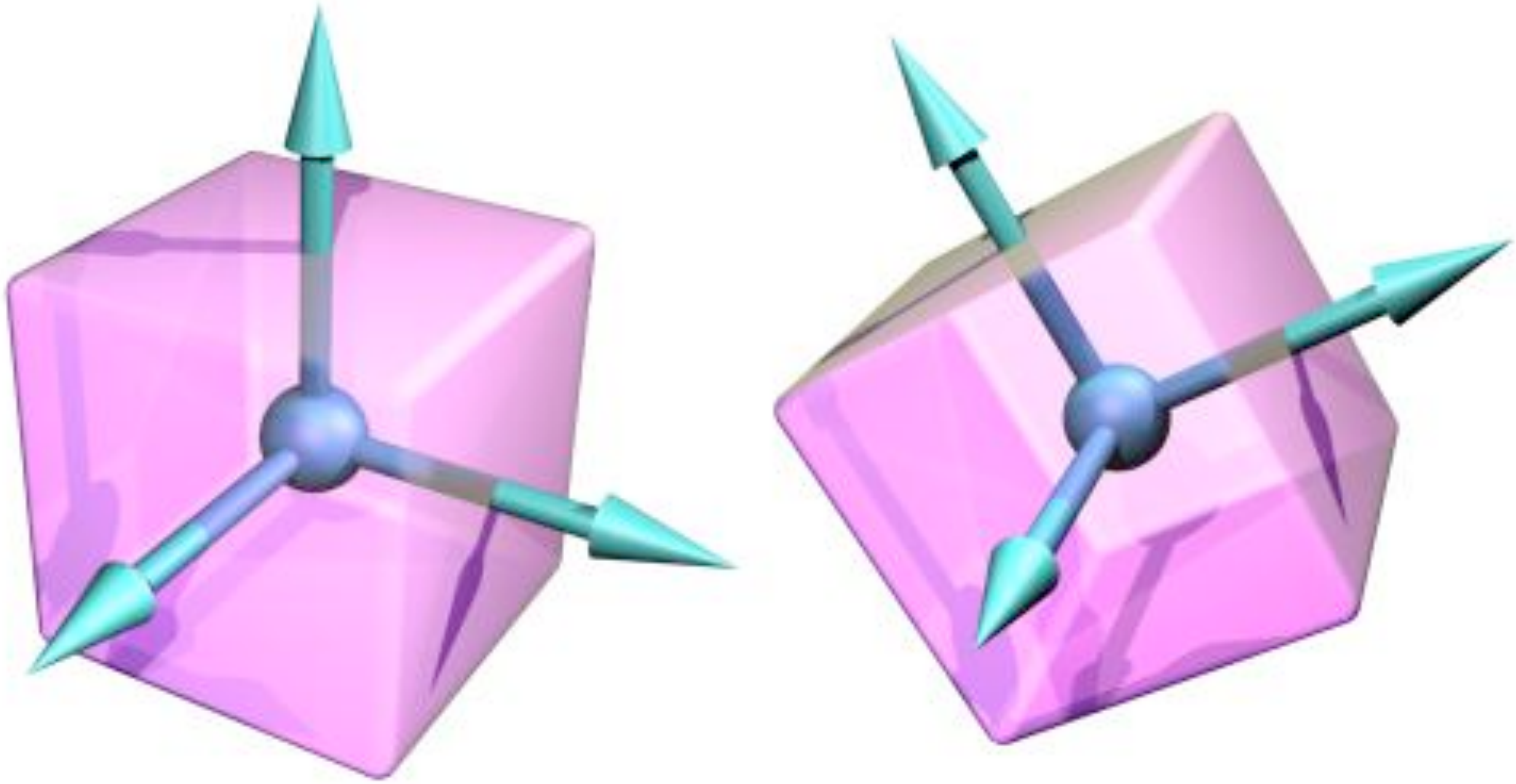


DOFs and Coordinate Spaces

- Each <u>body</u> has its own <u>coordinate system</u>

link

frame



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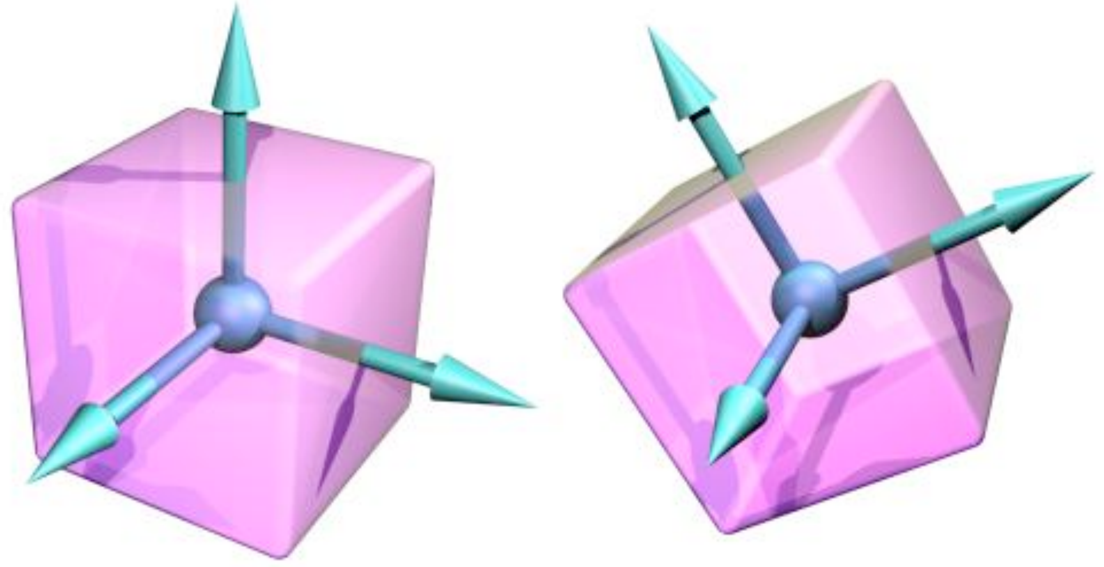
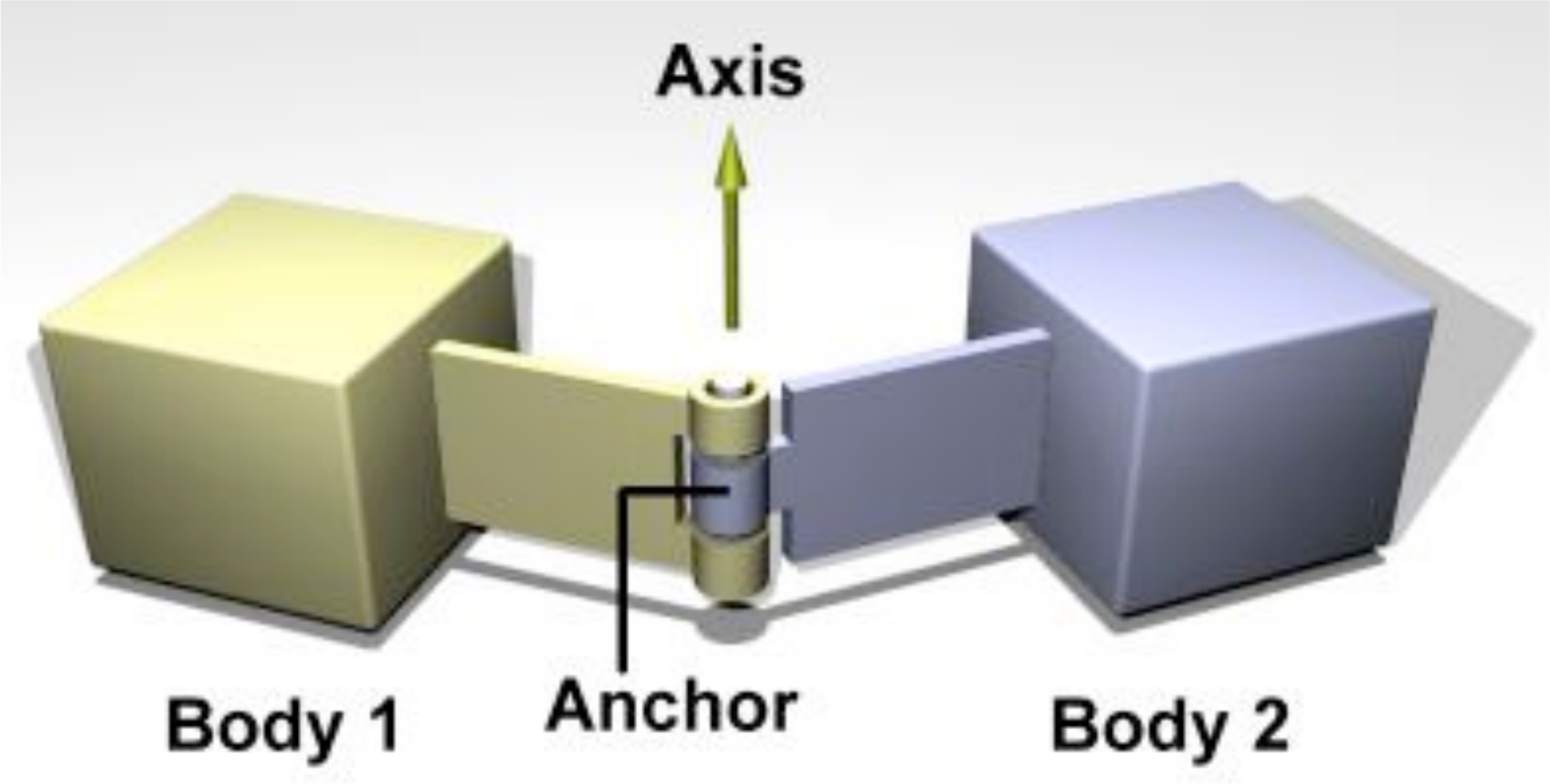


- Joints connect two links (rigid bodies)
  - Hinge (1 rotational DOF)

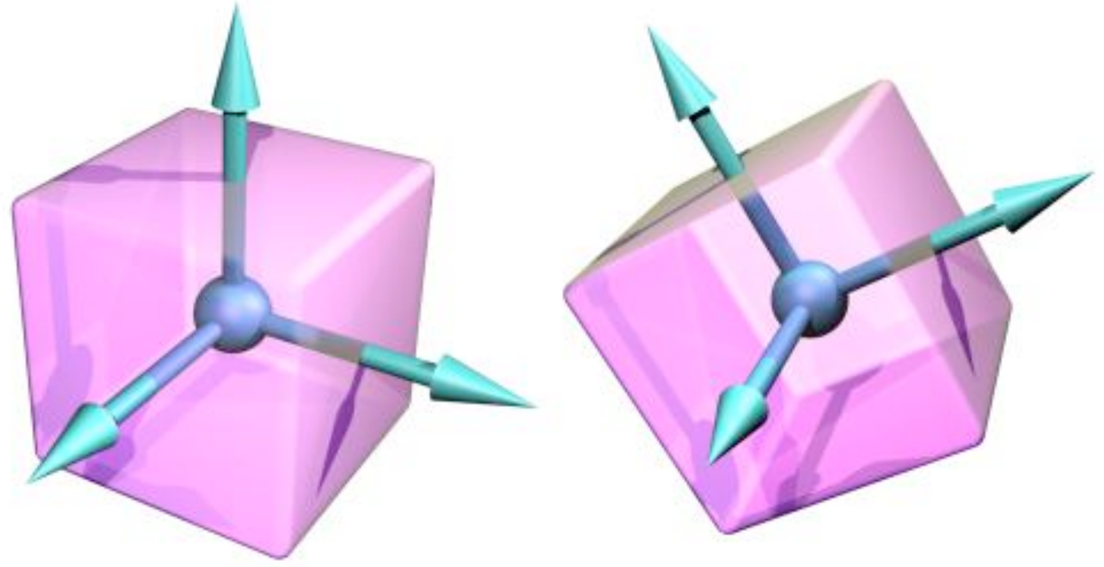


Hinge joint

DOFs and Coordinate Spaces

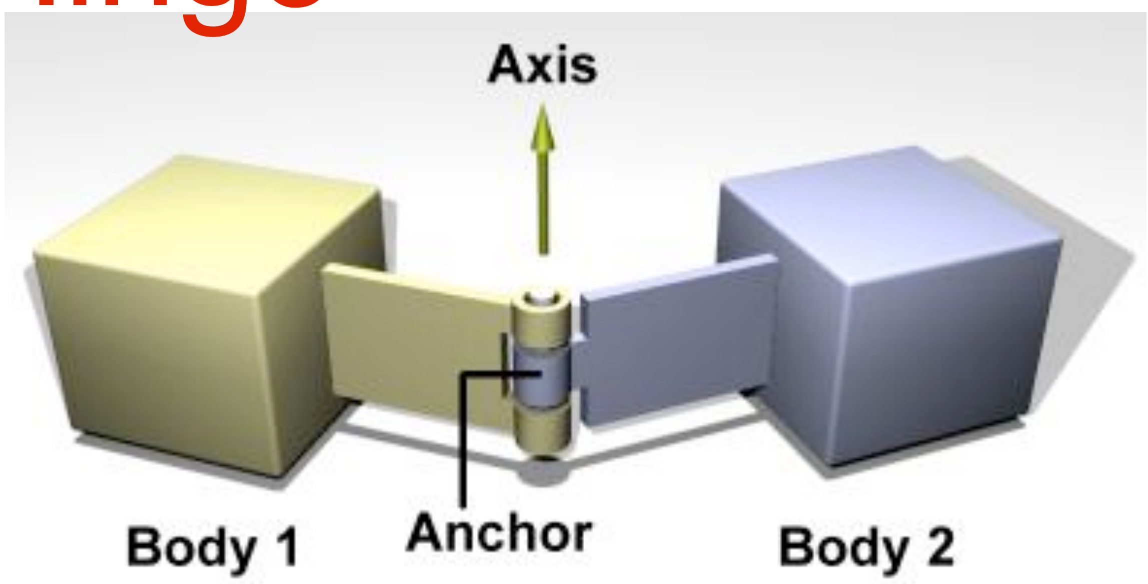
- Each <u>body</u> has its own <u>coordinate system</u>

link frame

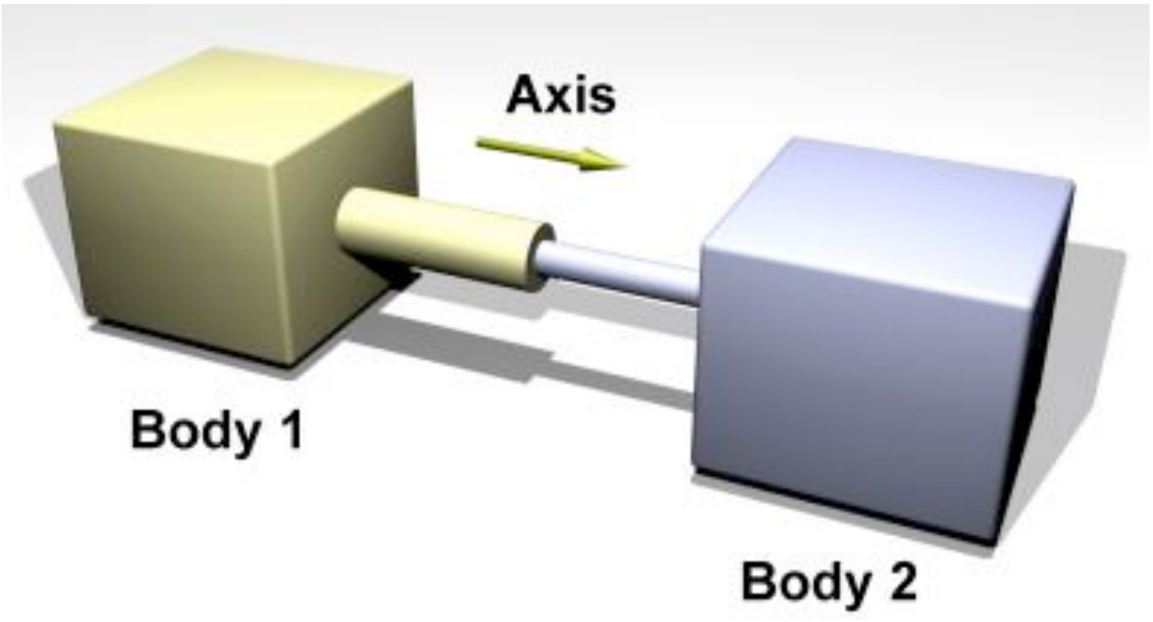


- Joints connect two links (rigid bodies)
  - Hinge (1 rotational DOF)
  - Prismatic (1 translational DOF)
  - Ball-socket (3 DOFs)

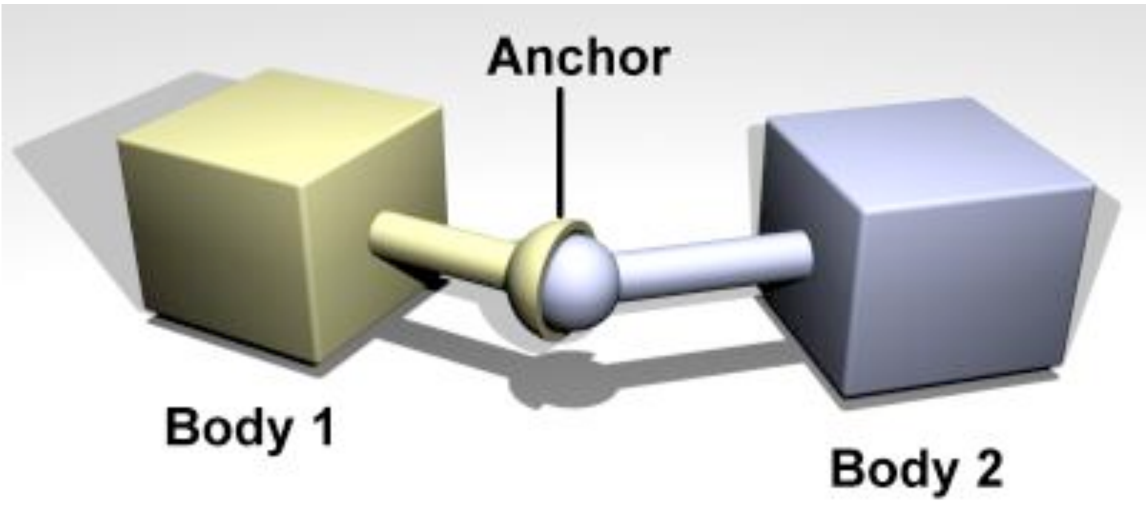
Hinge



Prismatic



Ball

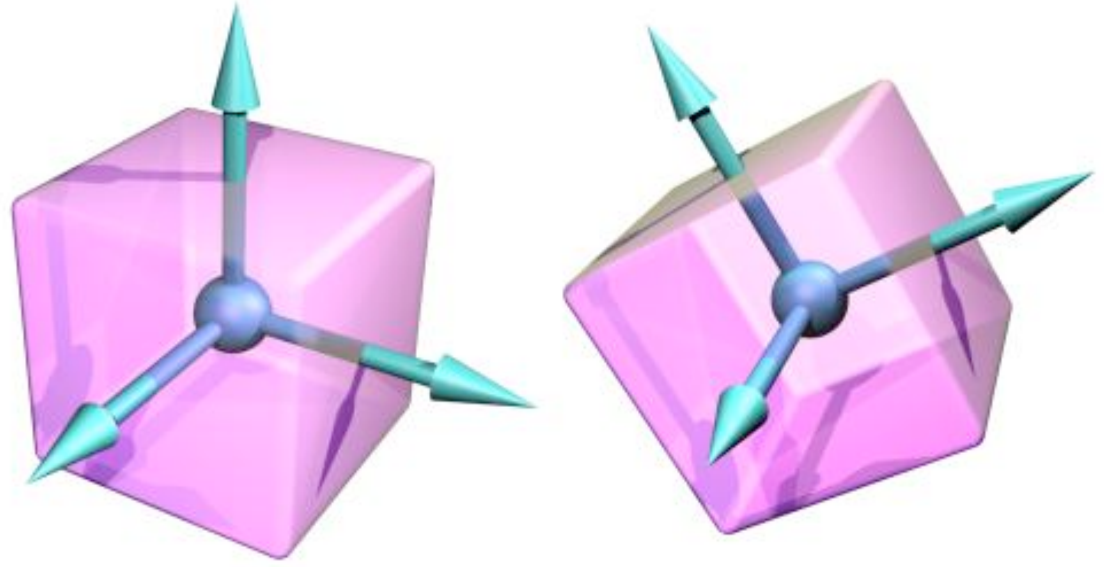


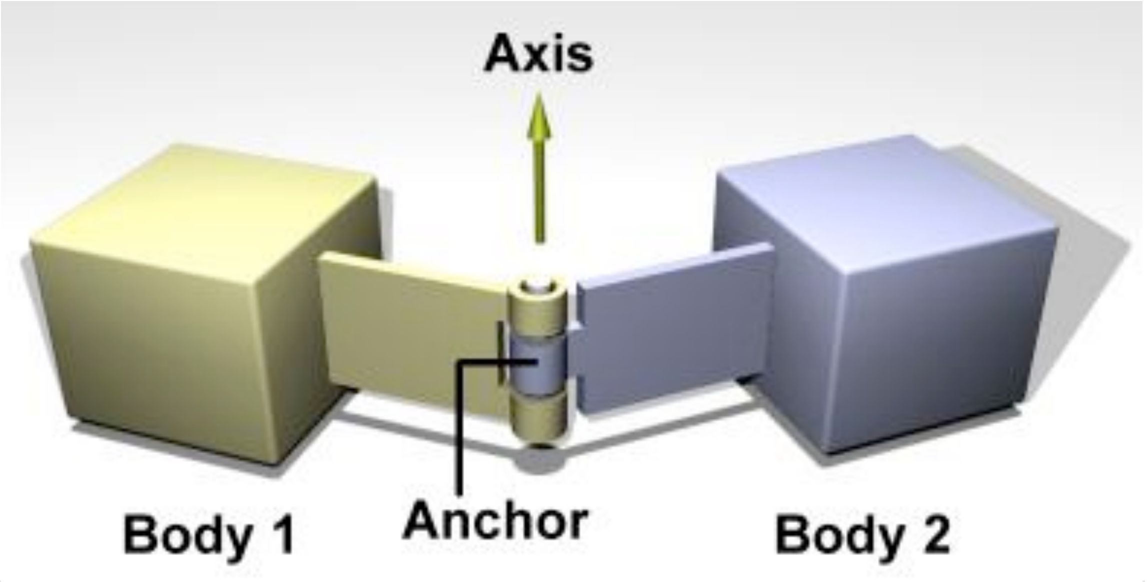
DOFs and Coordinate Spaces

- Each <u>body</u> has its own <u>coordinate system</u>

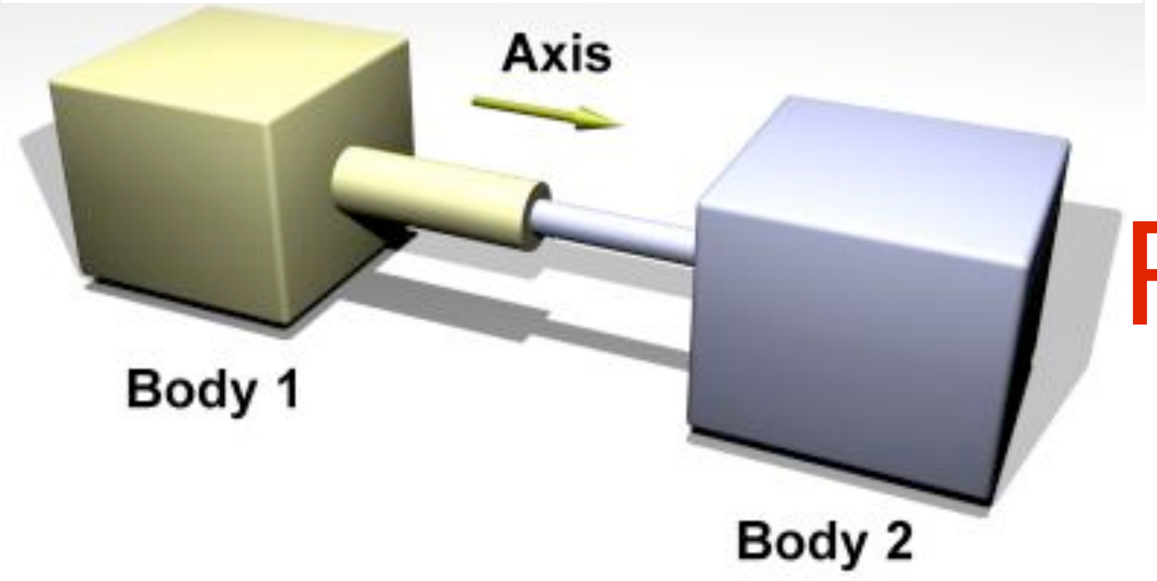
  link frame

- Joints connect two links (rigid bodies)
  - Hinge (1 rotational DOF)
  - Prismatic (1 translational DOF)
  - Ball-socket (3 DOFs)
- A motor exerts force along a DOF axis

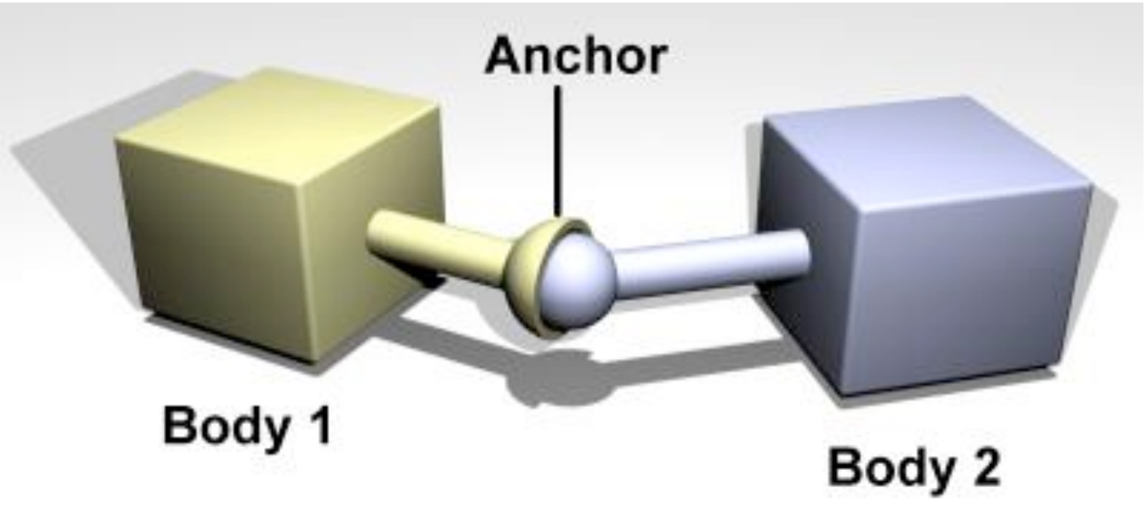




Hinge



Prismatic



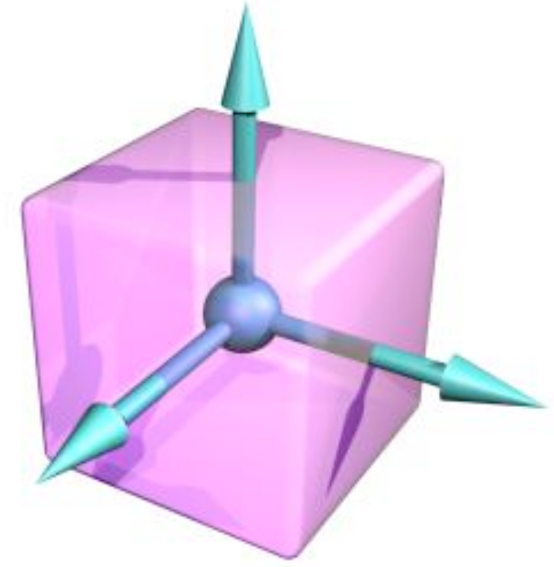
Ball

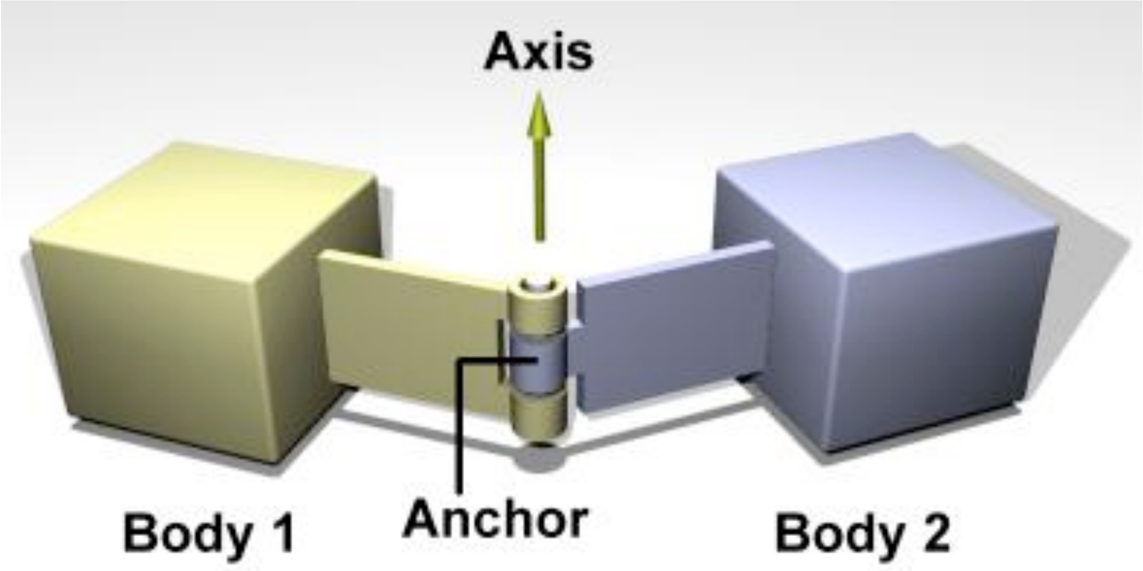
DOFs and Coordinate Spaces

- Each body has its own coordinate system

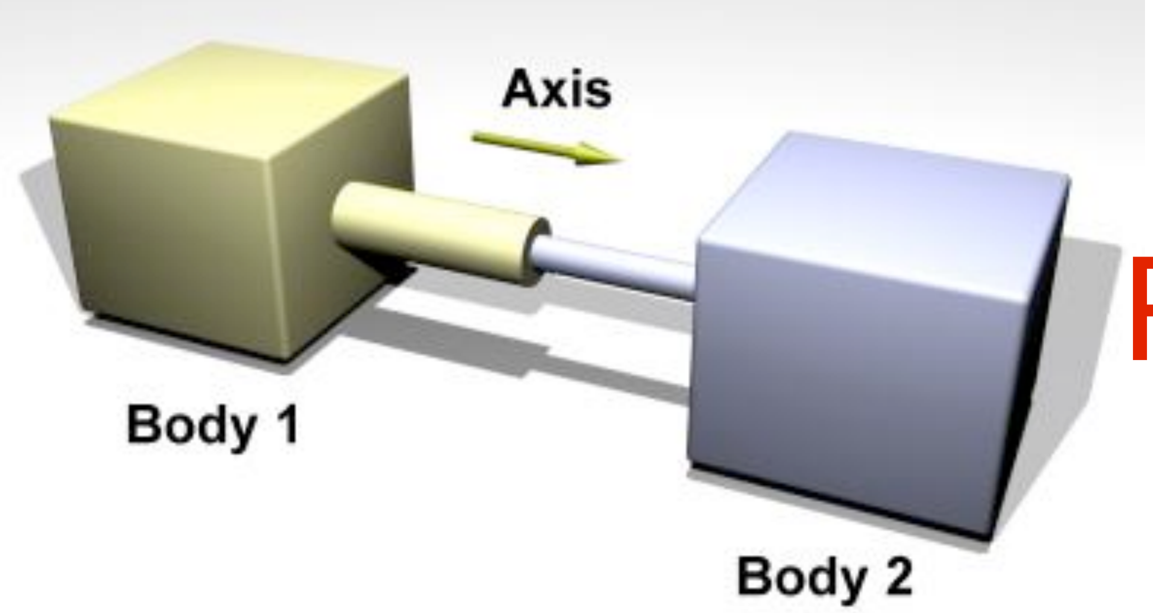
link frame

- Joints connect two links (rigid bodies)
  - Hinge (1 rotational DOF)
  - Prismatic (1 translational DOF)
  - Ball-socket (3 DOFs)
- A motor exerts force along a DOF axis
- Linear transformations used to relate coordinate frames of robot links and joints
- Spatial geometry attached to each link, but does not affect the body's coordinate frame

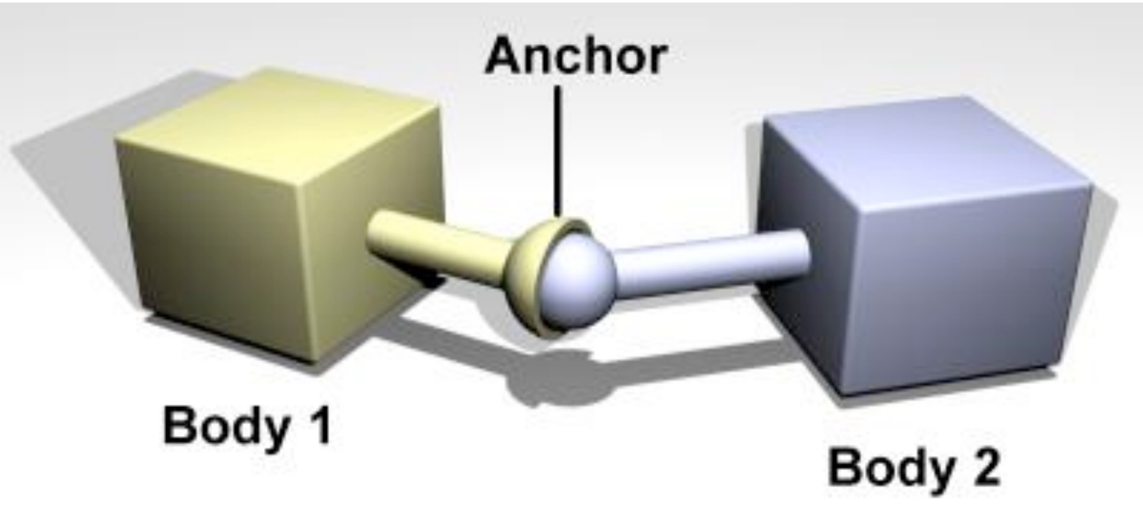




Hinge



Prismatic



Ball

Robotic machines are comprised of *N* joints and *N+1* links

Joints and links form a tree hierarchy of articulated motion as an "open chain"

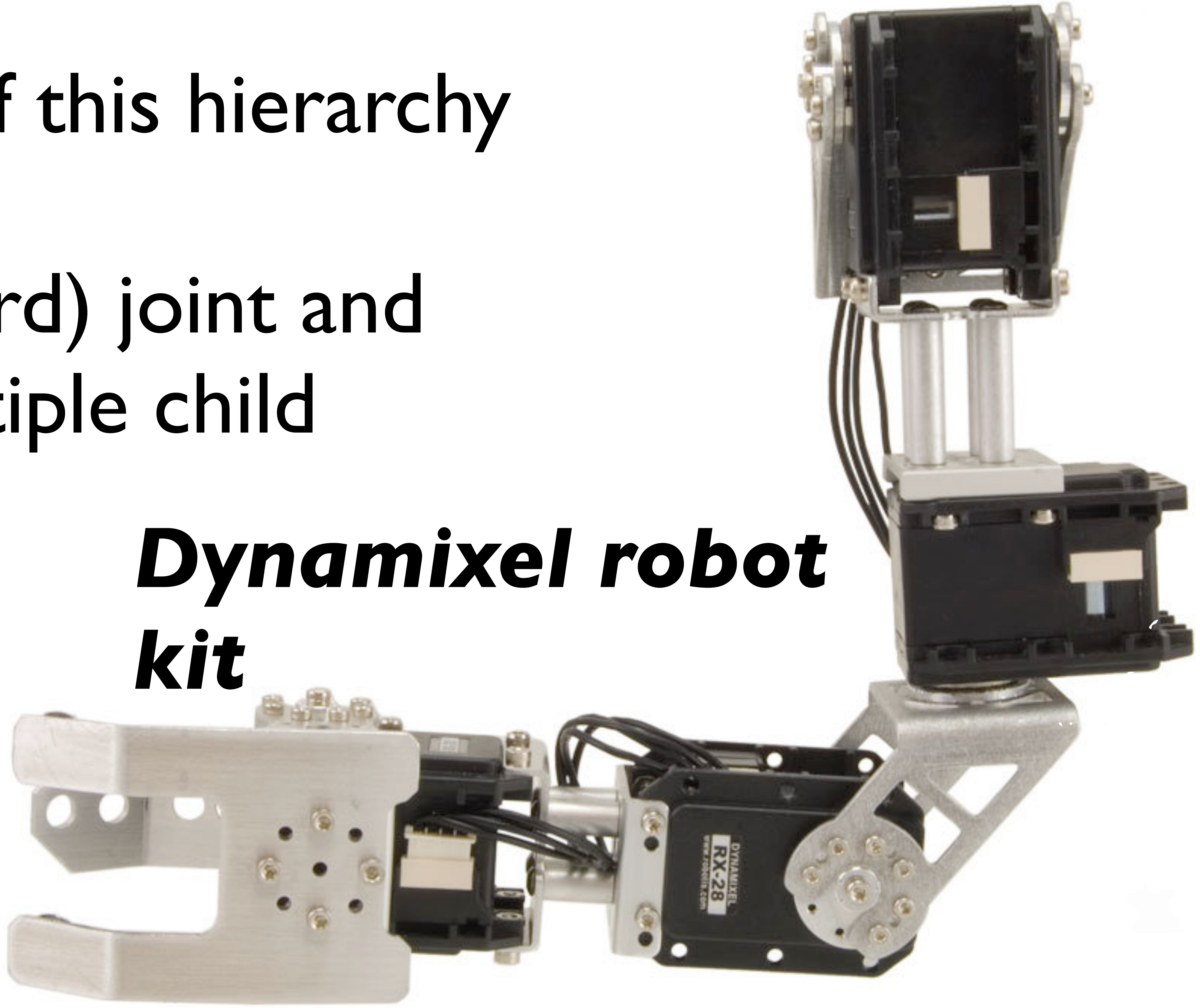
The "base" is the root link of this hierarchy

A link has one parent (inboard) joint and potentially zero, one, or multiple child (outboard) joints

A "serial chain" is a robot where every link has only one child joint

DLR Lightweight arm

Dynamixel robot kit



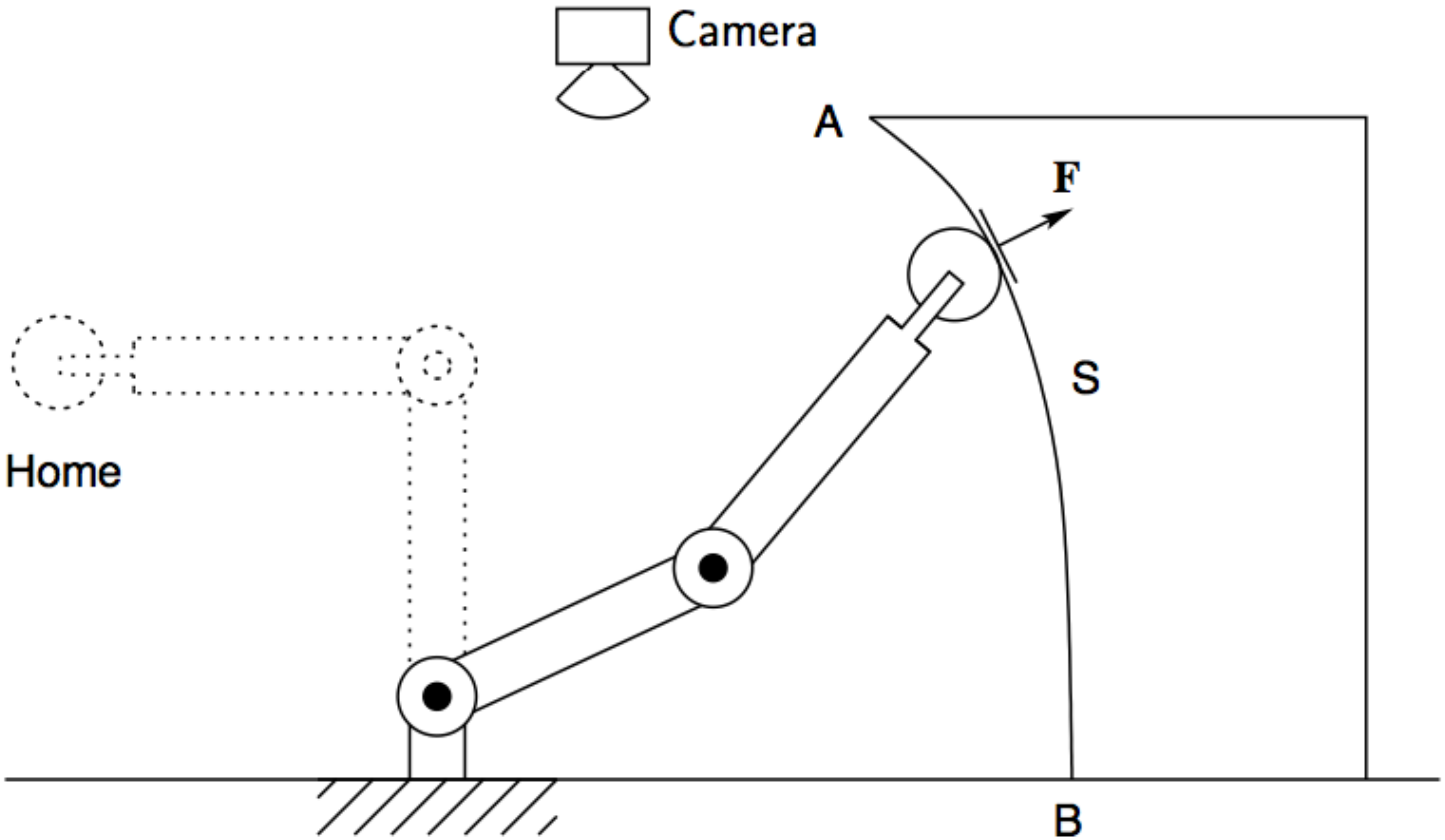
Consider some examples of robots…

Planar Arm

https://www.nowhereelse.fr/aikon-project-robot-artiste-dessin-portraits-29397/

AIKON-II Project - https://youtu.be/bbdQbyff_Sk

Planar 2-DOF 2-link Arm



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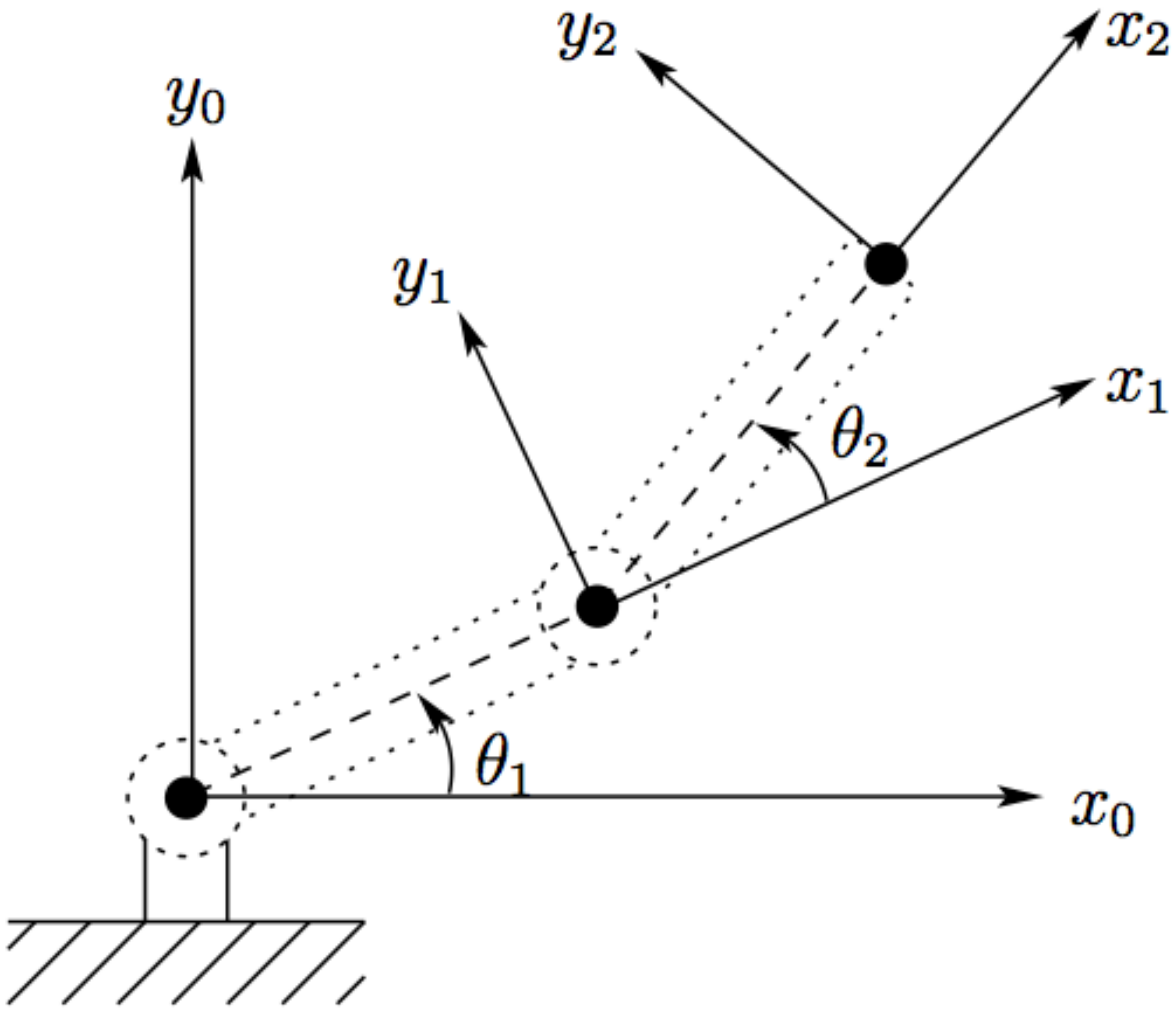

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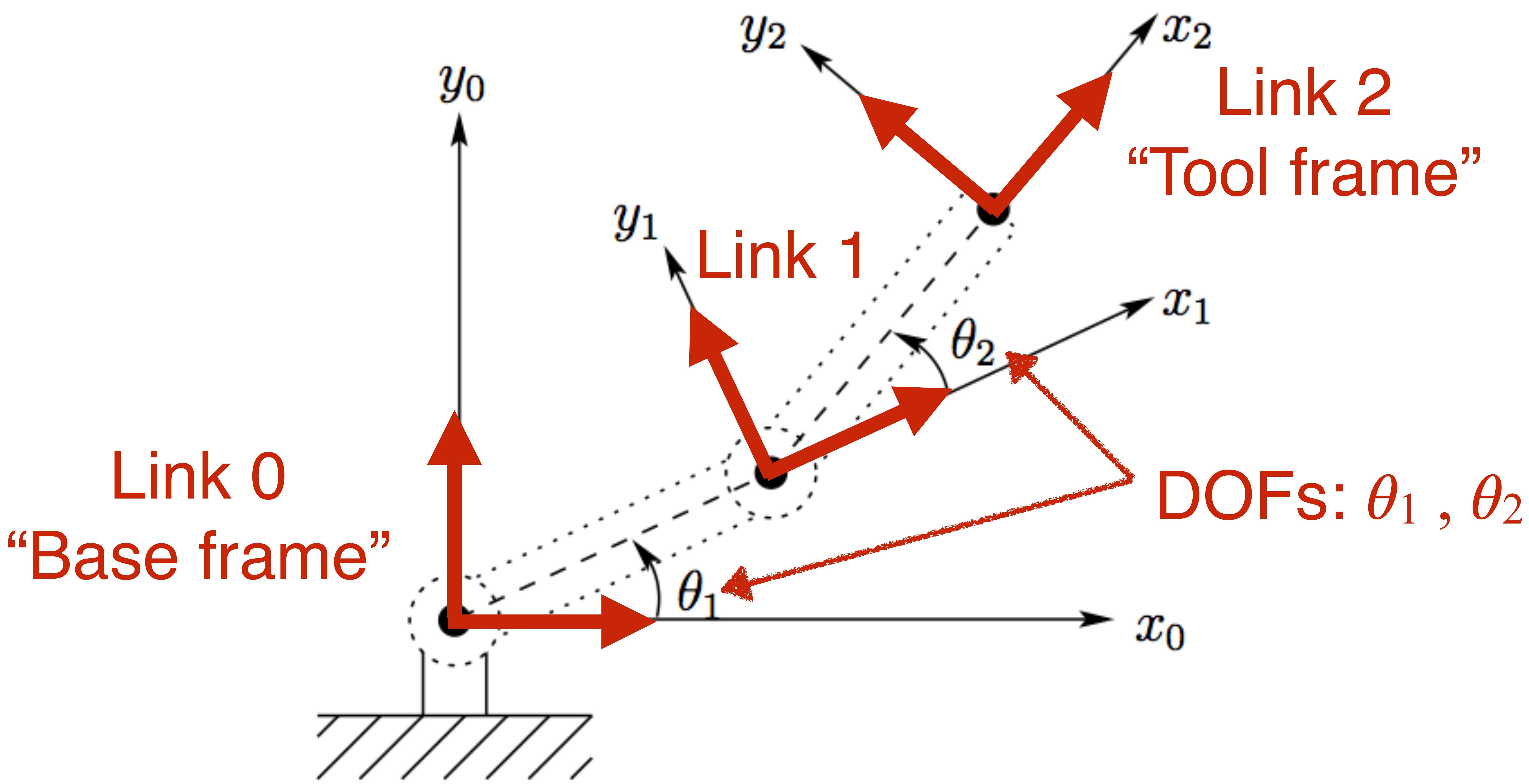
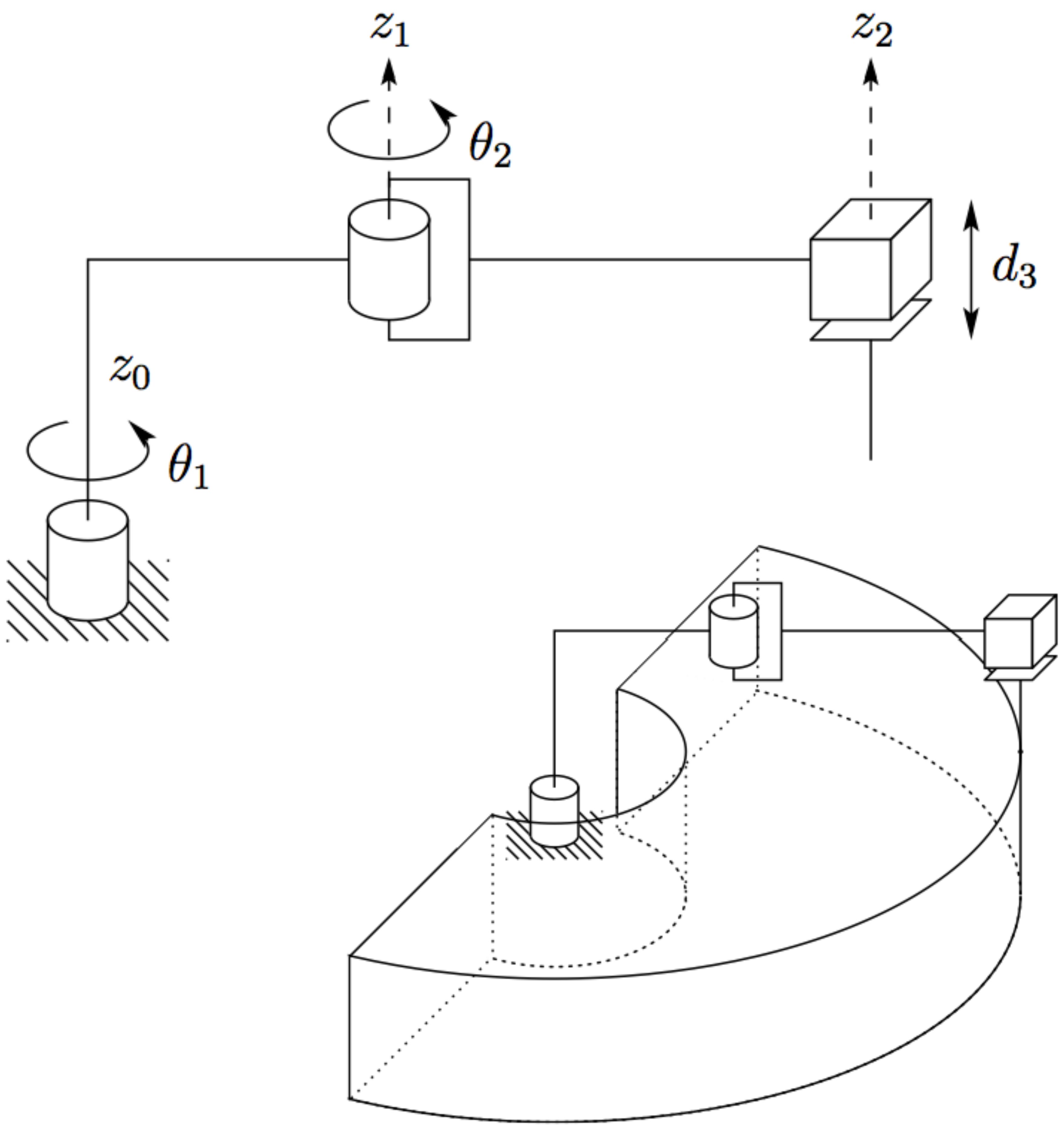
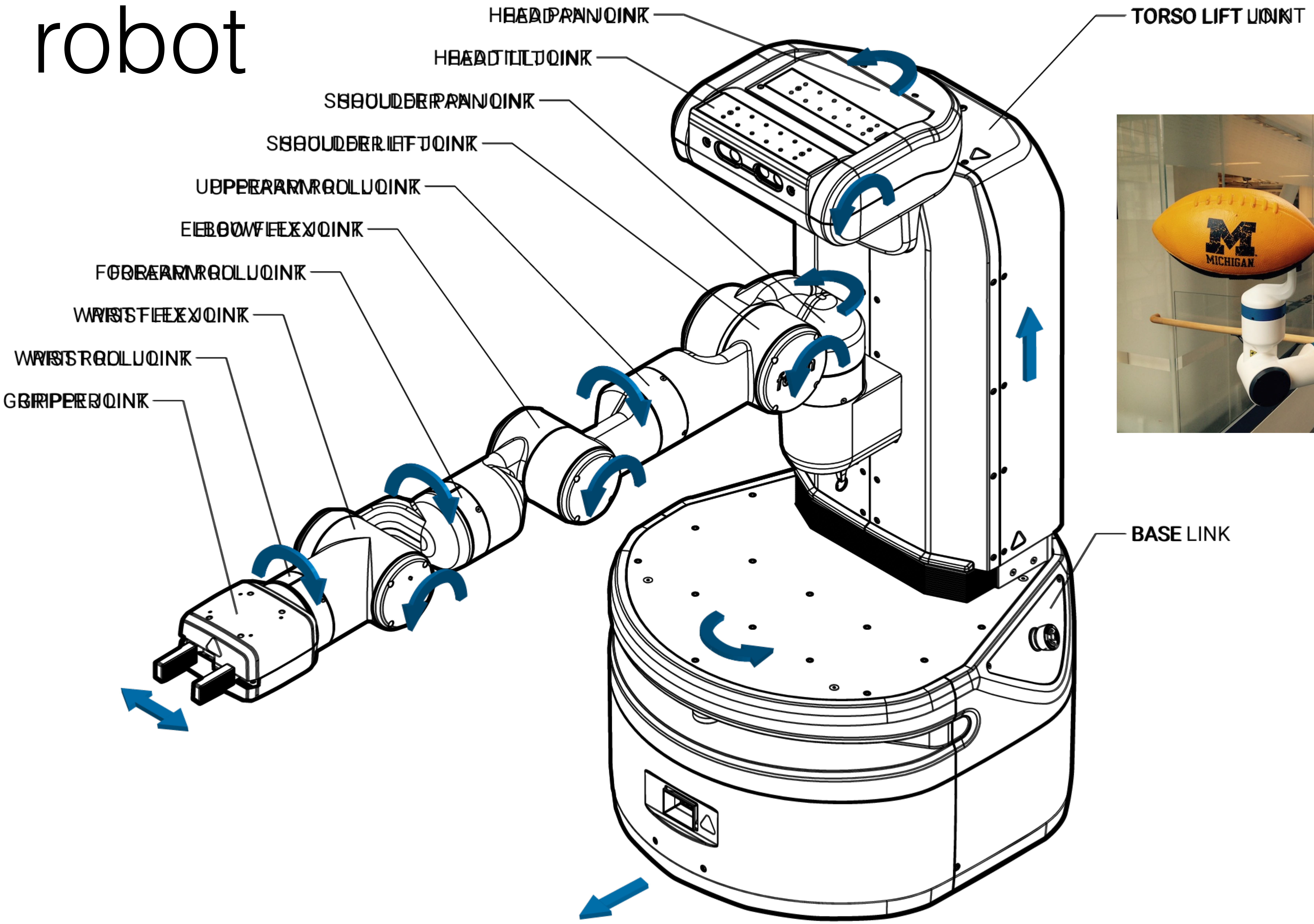


SCARA Arm

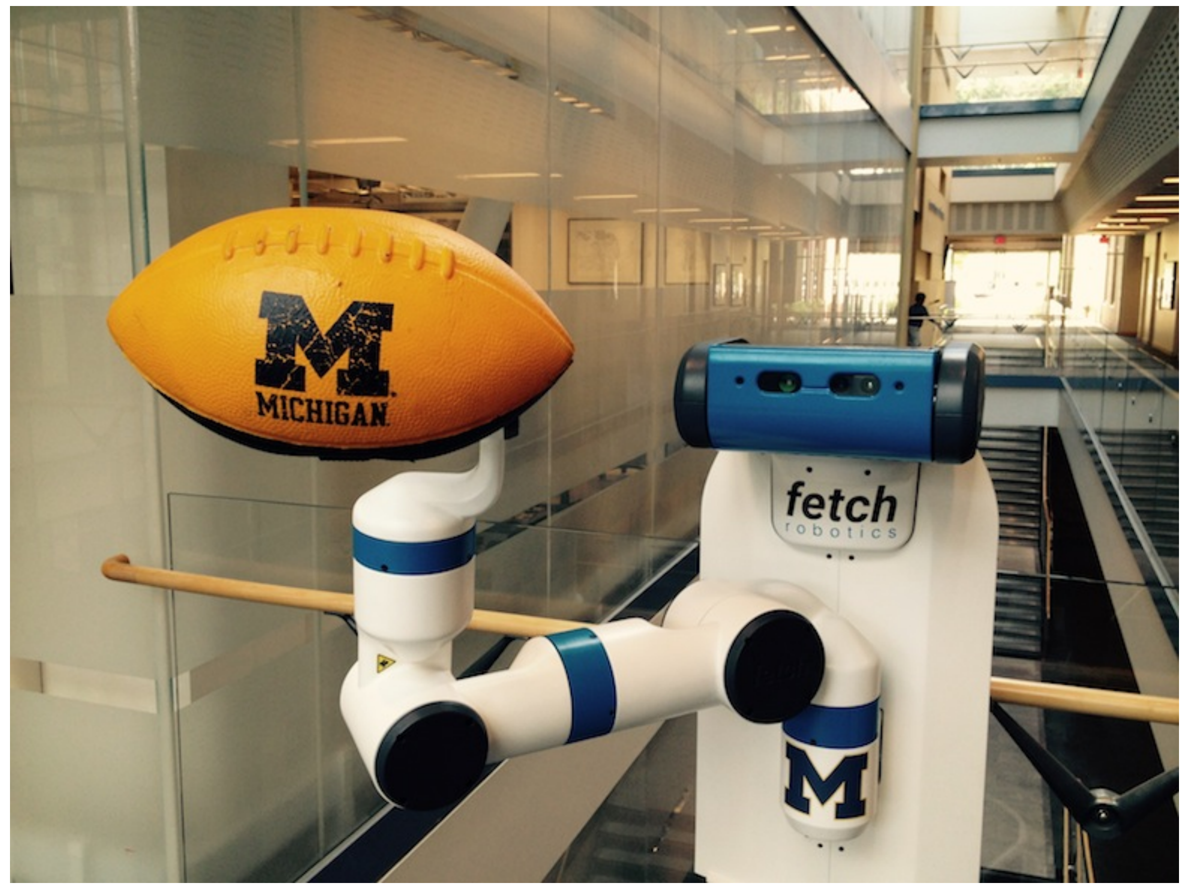
Selective Compliance Assembly Robot Arm



Fetch robot

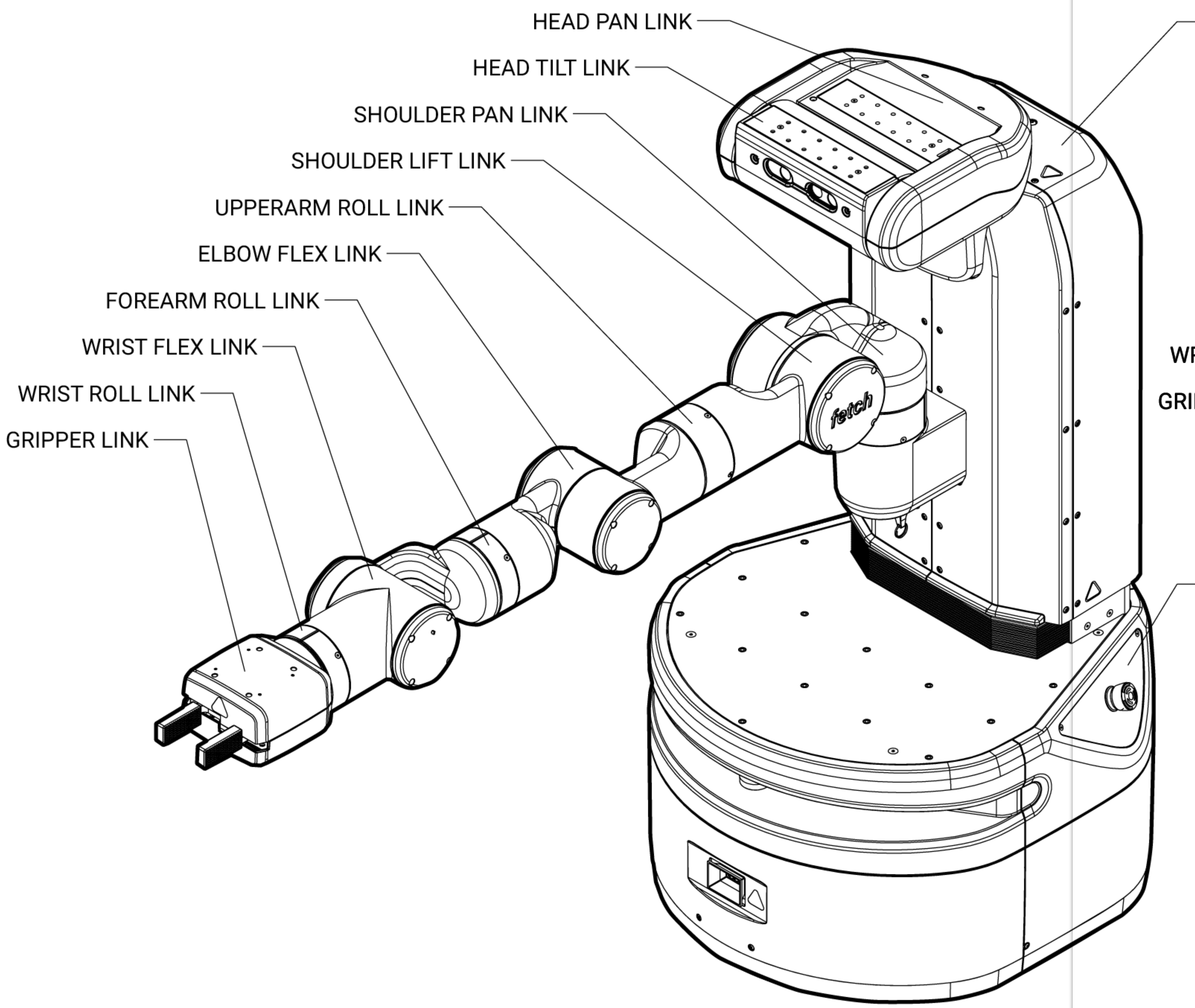


HEAD PAN JOINT
HEAD TILT JOINT
SHOULDER PAN JOINT
SHOULDER LIFT JOINT
UPPERARM ROLL JOINT
ELBOW FLEX JOINT
FOREARM ROLL JOINT
WRIST FLEX JOINT
WRIST ROLL JOINT
GRIPPER JOINT
TORSO LIFT JOINT
BASE LINK

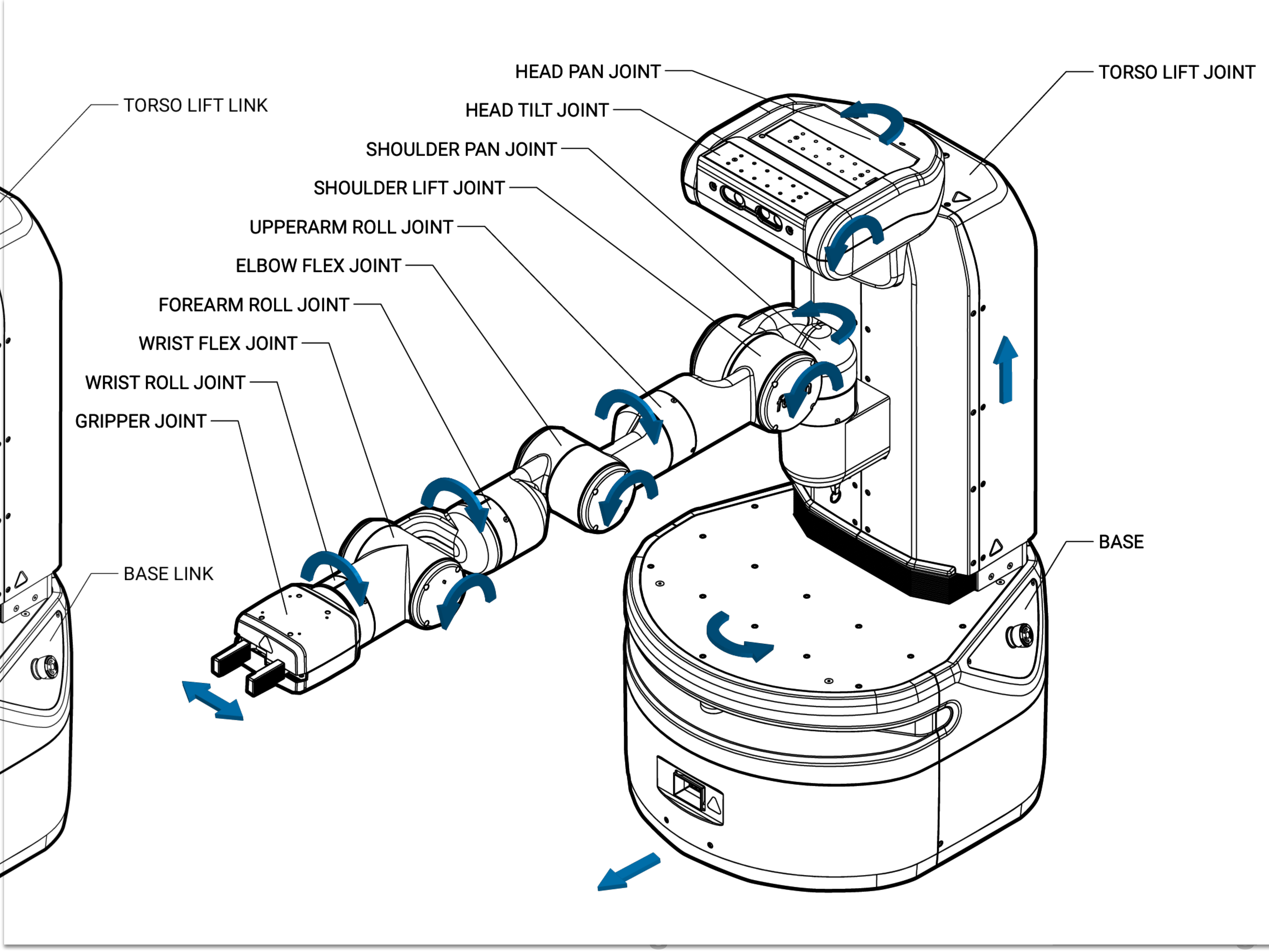


M
MICHIGAN
fetch

Fetch links



Fetch joints

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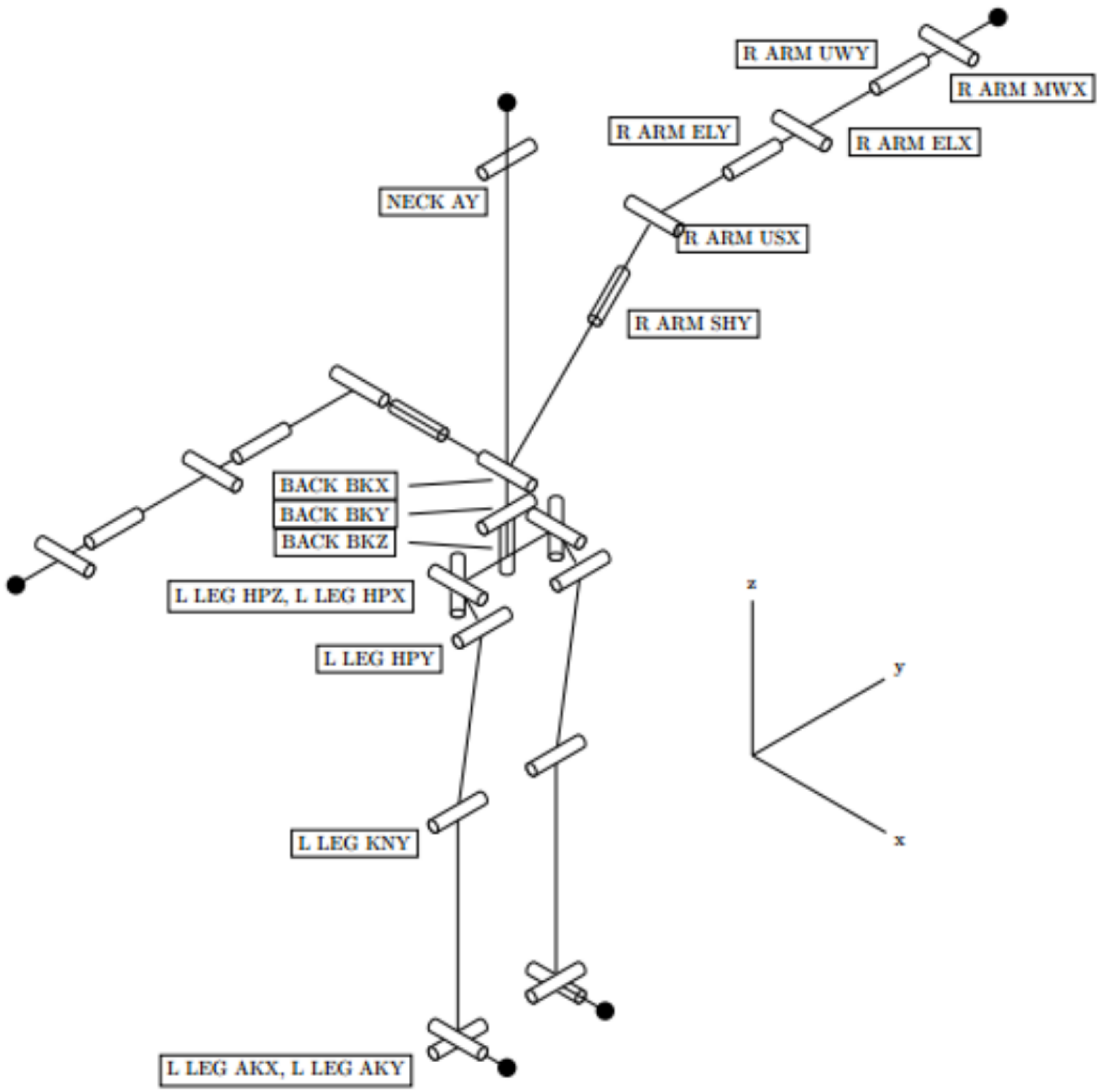
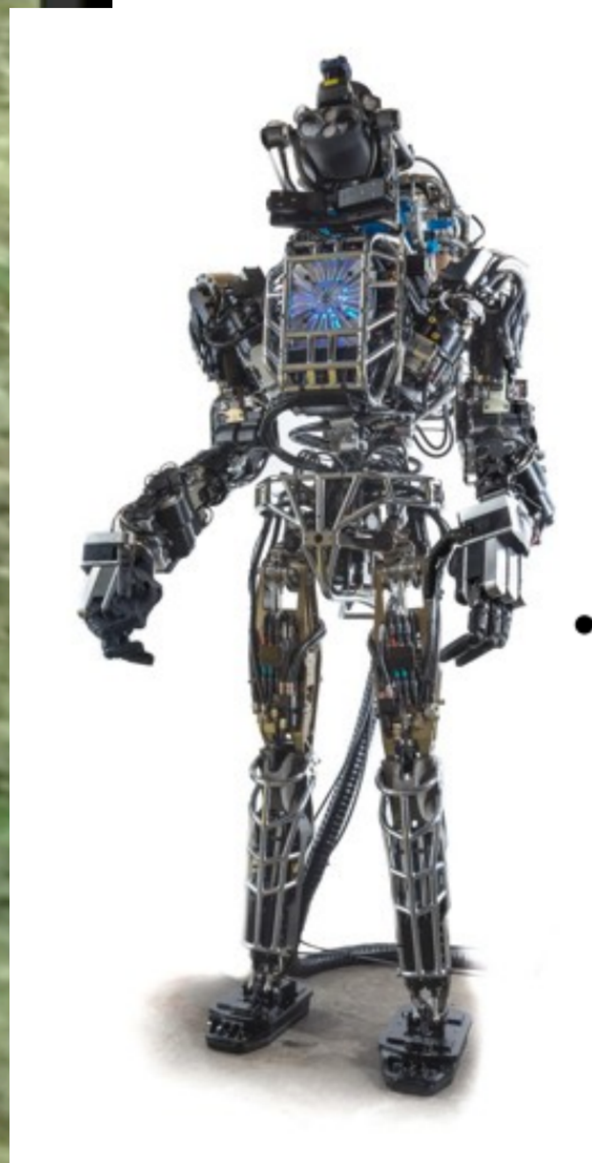
HEAD TILT JOINT
SHOULDER PAN JOINT
SHOULDER LIFT JOINT
UPPERARM ROLL JOINT
ELBOW FLEX JOINT
FOREARM ROLL JOINT
WRIST FLEX JOINT
WRIST ROLL JOINT
GRIPPER JOINT
TORSO LIFT JOINT
BASE
BOXED WATER IS BETTER.

Atlas robot



WPI

Physical Simulation (Gazebo)



Why simulate robots?

Why simulate robots?

- **Real robots are expensive**
  - Improper controllers can physically break robots
  - Inexpensive to experiment and test robot controllers in simulation
- **Predictive model of dynamics**
  - Necessary for some types of control (e.g., optimal control)

$$J = \Phi\left[\,\mathbf{x}(t_0), t_0, \mathbf{x}(t_f), t_f\,\right] + \boxed{\int_{t_0}^{t_f} \mathcal{L}\left[\,\mathbf{x}(t), \mathbf{u}(t), t\,\right] \mathrm{d}\,t}$$

Why simulate robots?

- **Robot simulation packages are readily available**

GAZEBO

DART
Dynamic Animation and Robotics Toolkit

Webots
robot simulation

- **Built on physics engines that numerically integrate over dynamics**

SD/FAST
(the OG)

vortex
studio

newton
DYNAMICS

OPEN DYNAMICS ENGINE™

BULLET
PHYSICS LIBRARY

For example,
NASA Valkyrie…

Valkyrie for real - https://youtu.be/5Ee5u2ekE8c

NASA Valkyrie…
development in simulation

Valkyrie in Gazebo simulation - https://youtu.be/tLCpJvqgtRQ

Our first robot:

Pendularm!

Project 2: 1 DOF Pendularm

file:///Users/logan/git_tmp/kineval/pendularm/pendularm1.html

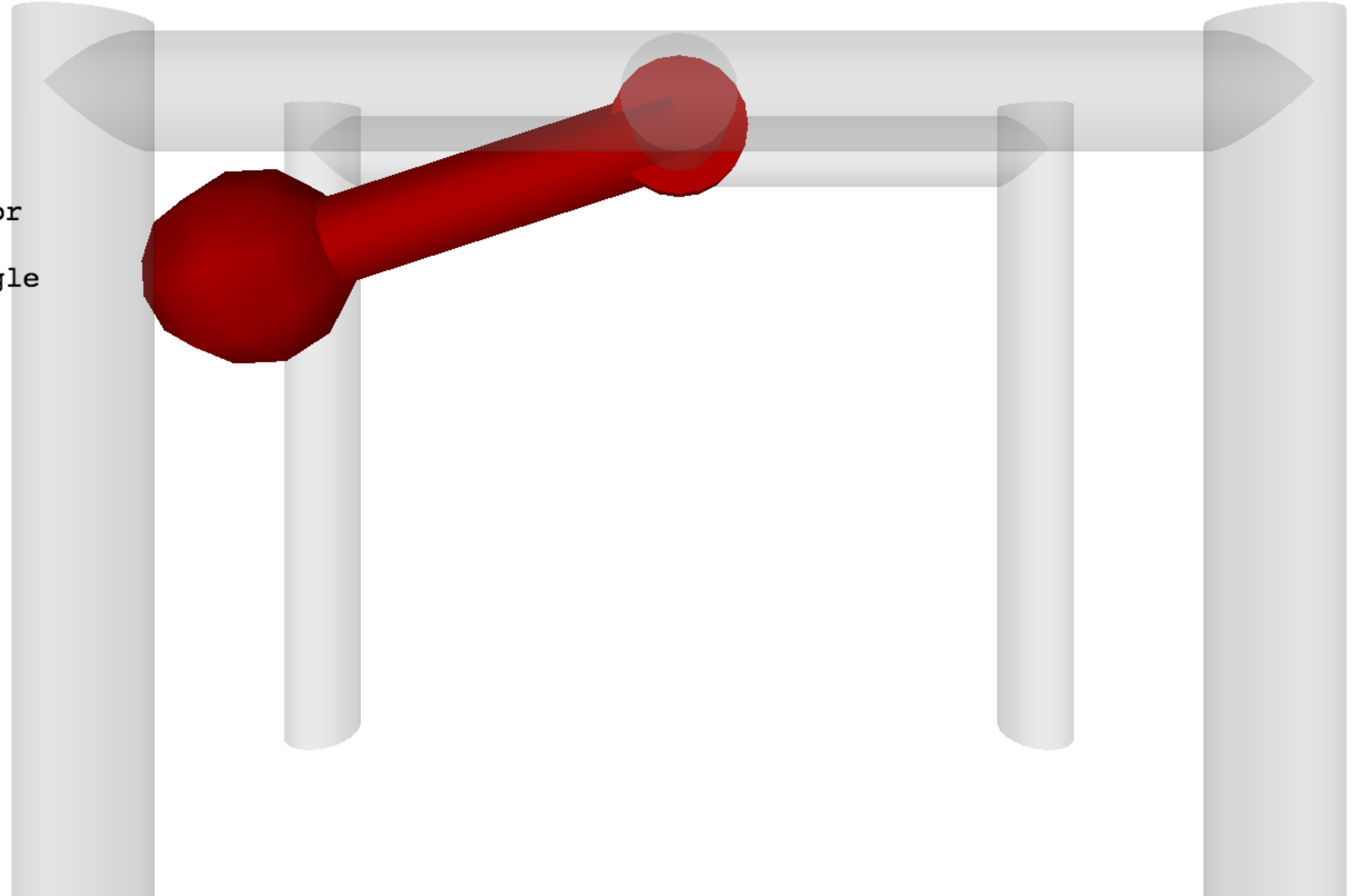
```
System
t = 162.00 dt = 0.05
integrator = velocity verlet
x = -1.26
x_dot = -0.00
x_desired = -1.26

Servo: active
u = -37.32
kp = 1500.00
kd = 15.00
ki = 150.10

Pendulum
mass = 2.00
length = 2.00
gravity = 9.81

Keys
[0-4] - select integrator
a/d - apply user force
q/e - adjust desired angle
c - toggle servo
s - disable servo
```

model, simulate, and control
1 DoF robot arm

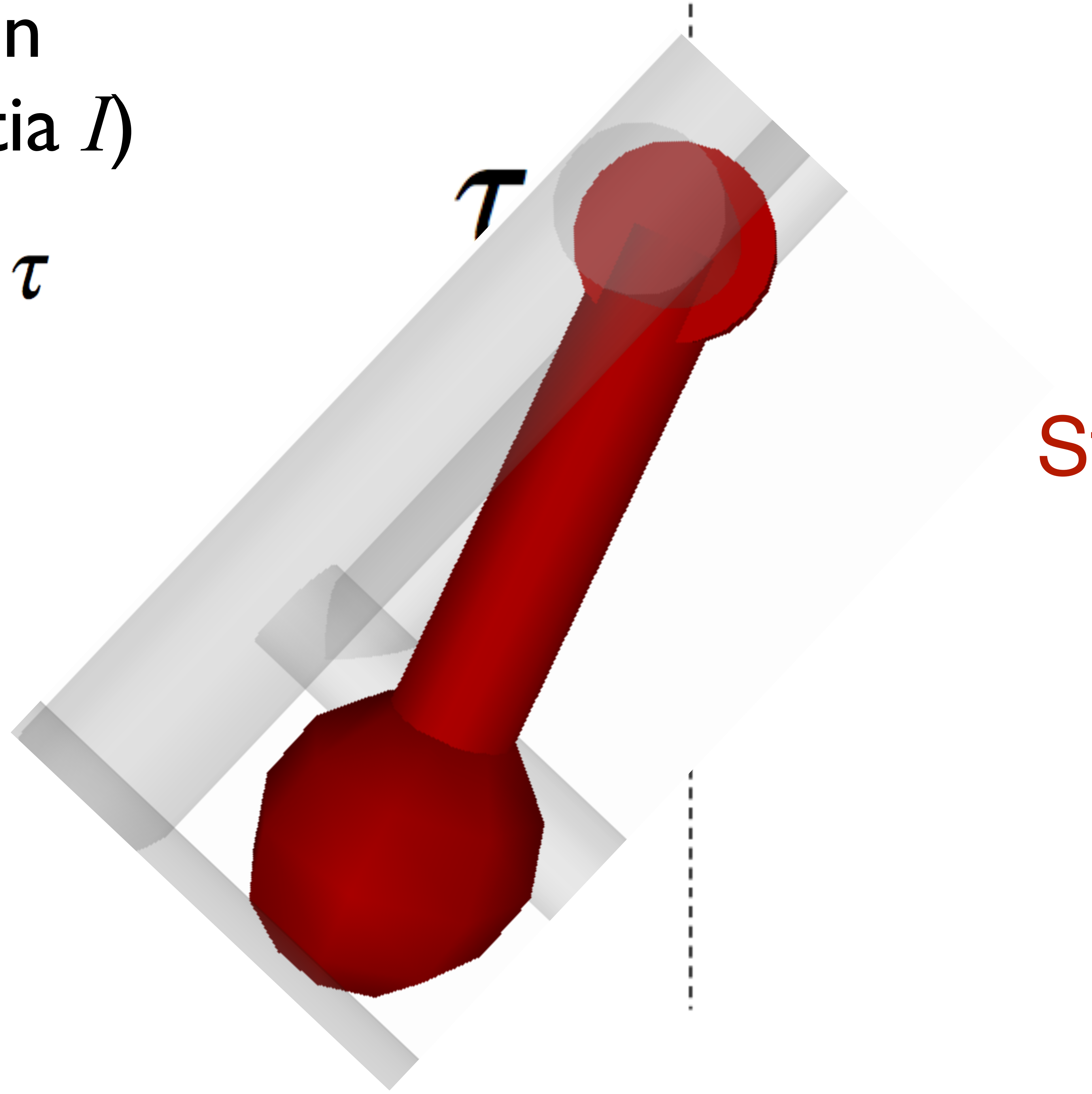


Example: Pendulum

Dynamics

Equation of motion
(with rotational inertia I)

$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$



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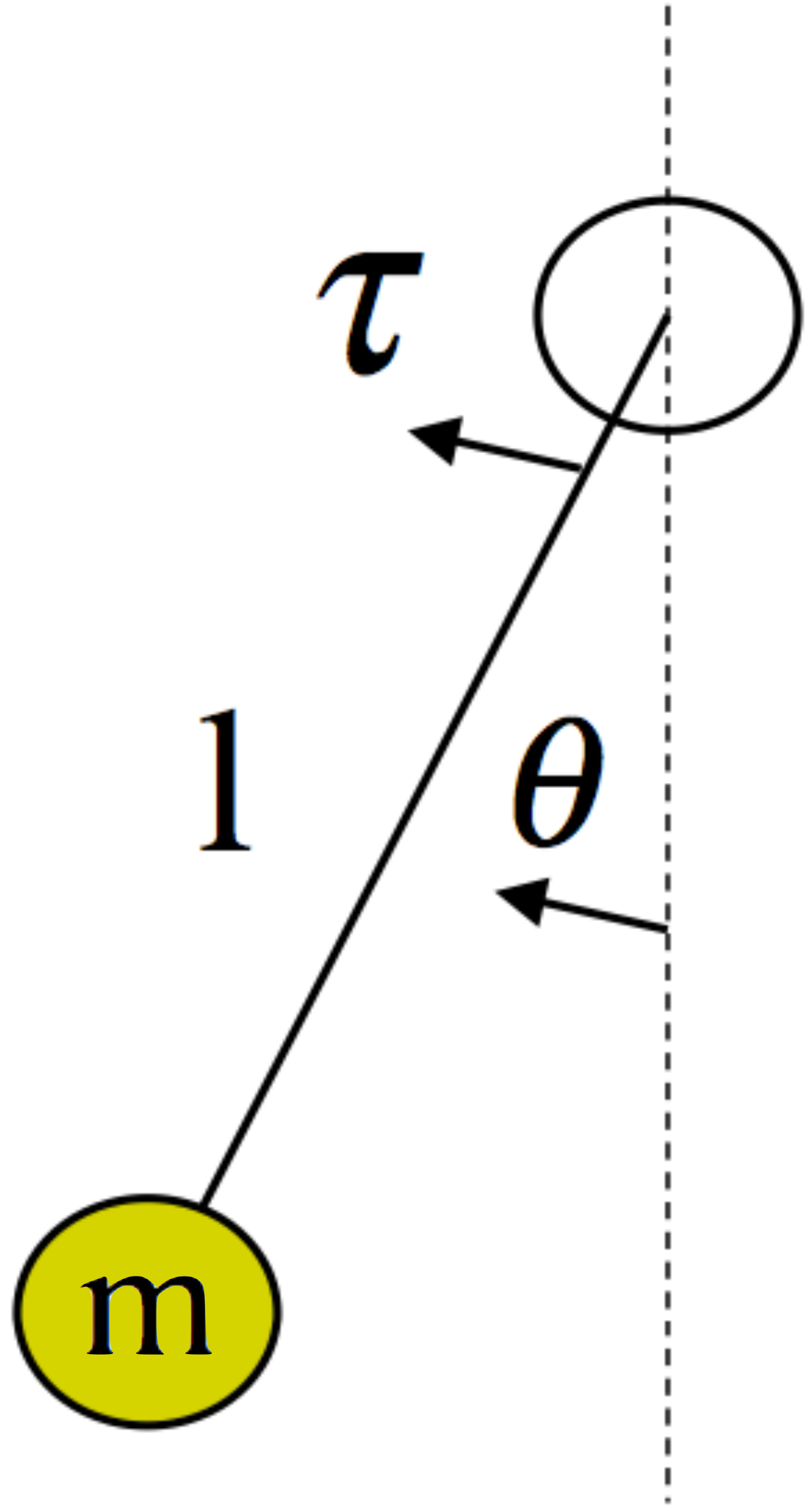
Pendulum of length *l* with
point mass *m*

Example: Pendulum

Dynamics

Equation of motion
(with rotational inertia I)

$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$



Controls

Motor produces torque
(angular force)

State (or Configuration)

Angle expresses pendulum
range of motion

System

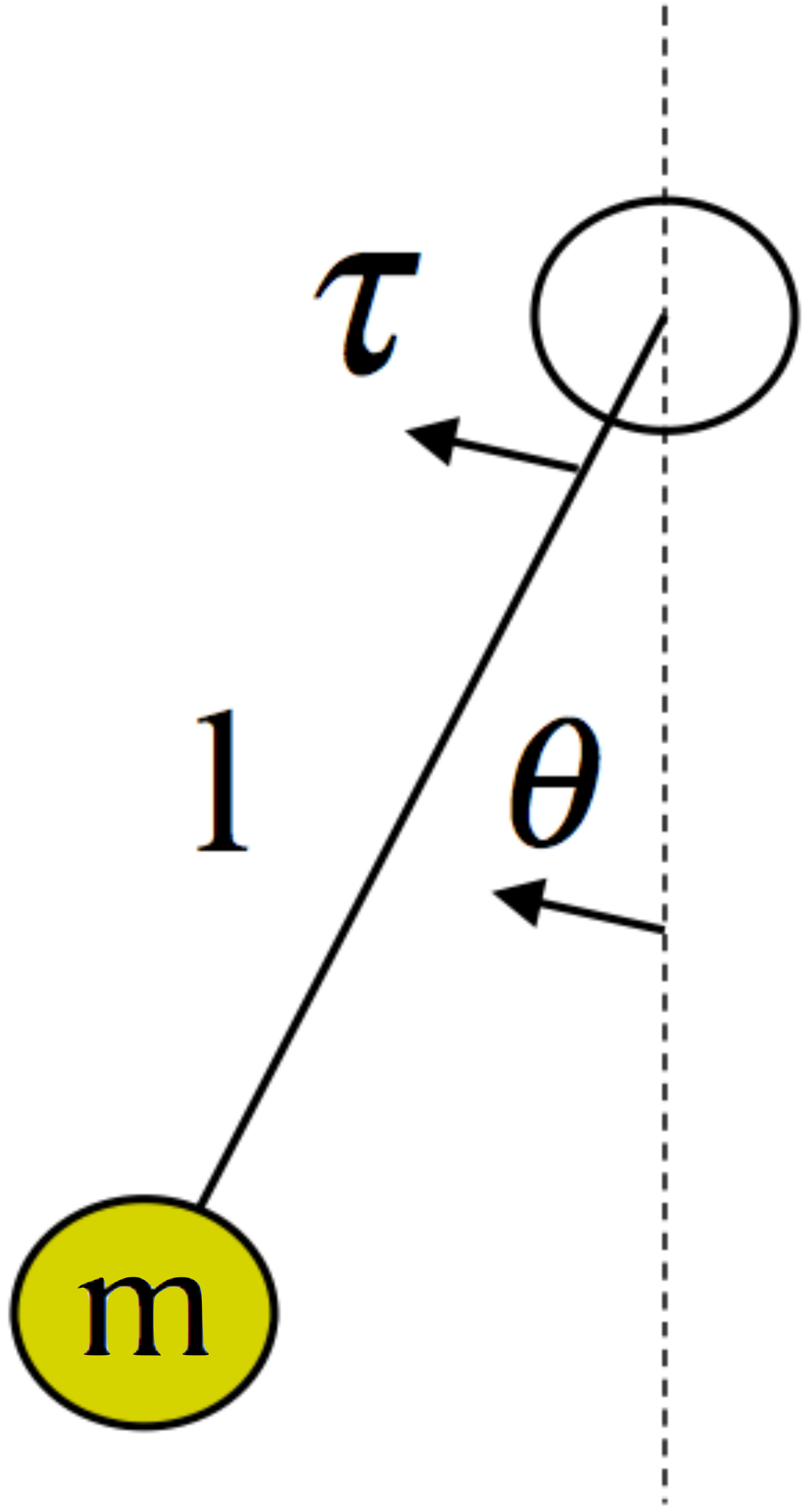
Pendulum of length *l* with
point mass *m*

Example: Pendulum

Equation of motion
(with rotational inertia I)

$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$

motor
torque



Motor produces torque
(angular force)

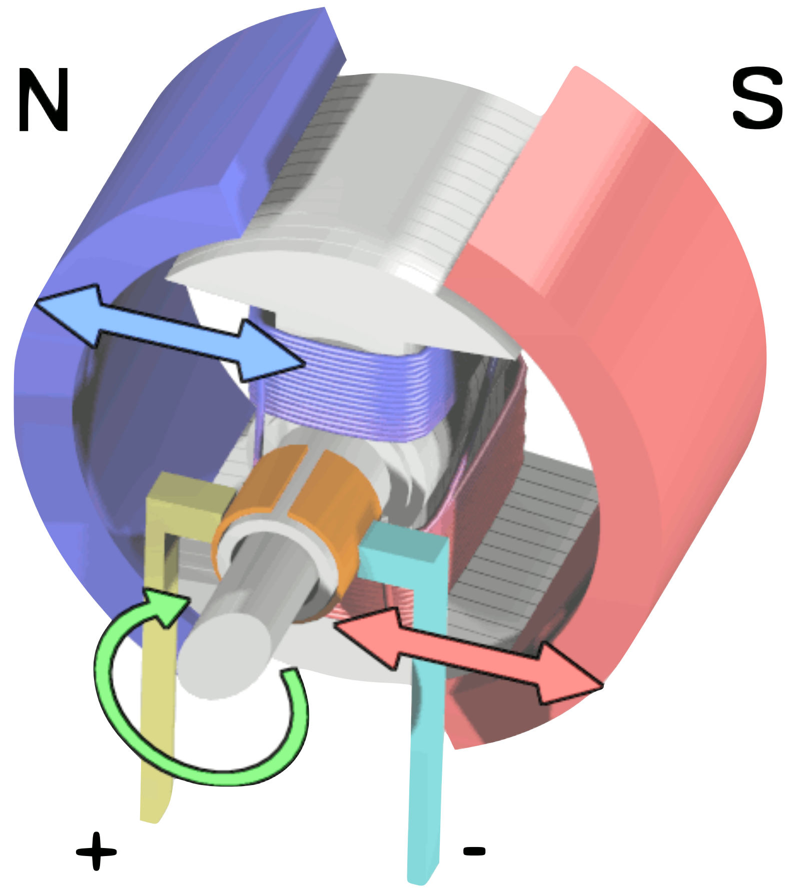
Angle expresses pendulum
range of motion

Pendulum of length l with
point mass m

Project #628
Stepper Motor
OBJECTIVE: To build a variable stepper motor.
Adjust the adjustable resistor (RV) to the middle position and turn on the slide switch (S1). As the circuit oscillates, the motor (M1) moves a short distance as the speaker (SP) sounds. Adjust the adjustable resistor to different positions seeing how it affects the motor and speaker.
Moving parts. Do not touch the fan or
Project #632
RV
PNP
Q1
100uF
C4
OFF
ON
S1
U2
ALARM IC
SPEAKER
D7
U1
MUSIC IC
AA
amazonbasics
ALKALINE

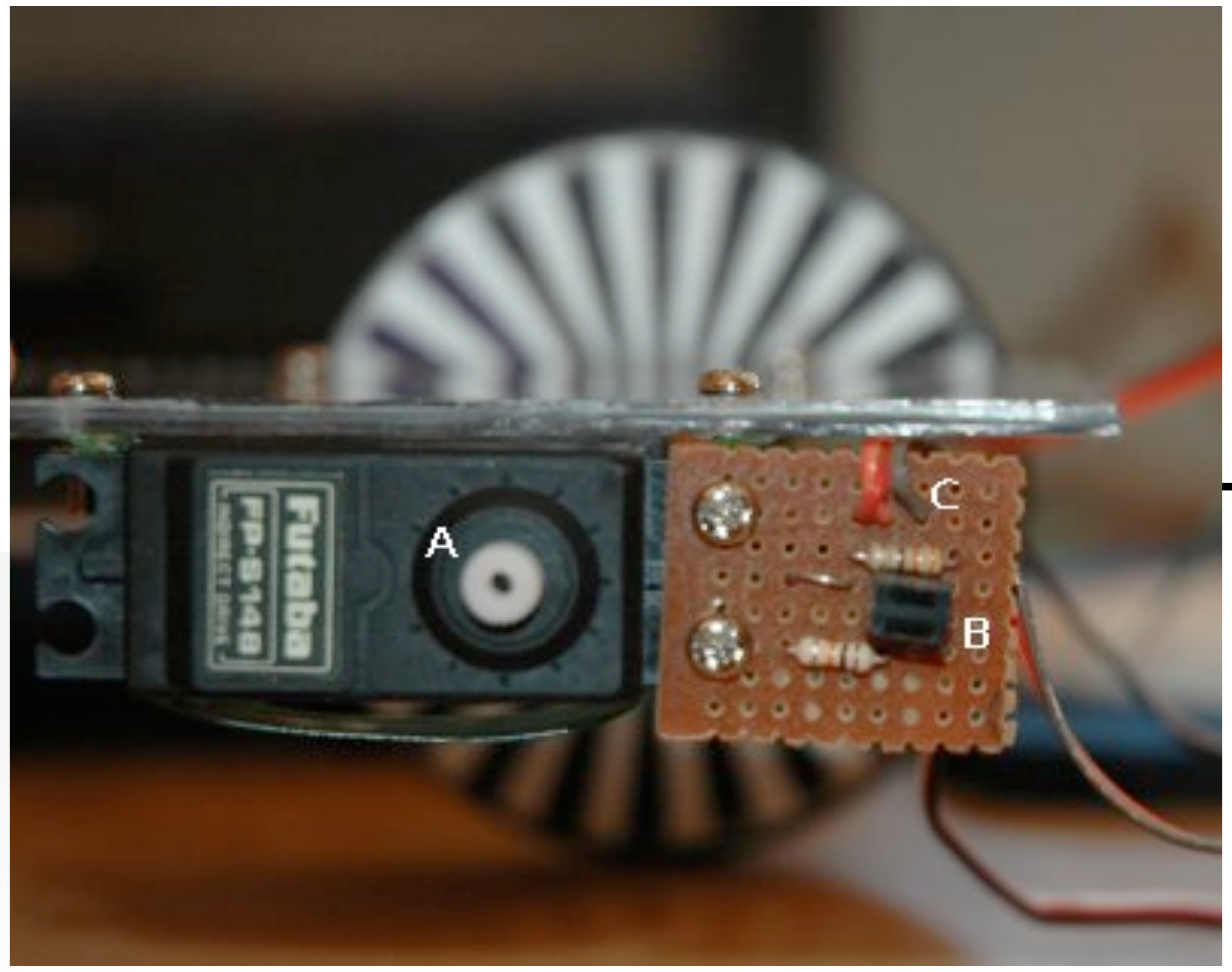
Actuator
(electric)

Actuator
(electric)

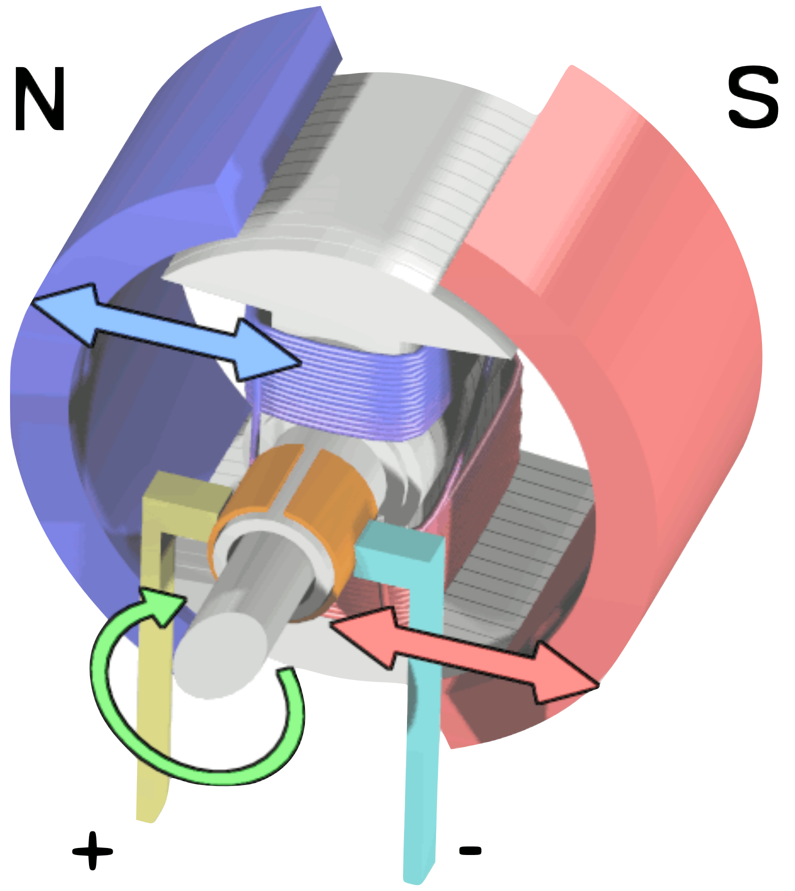


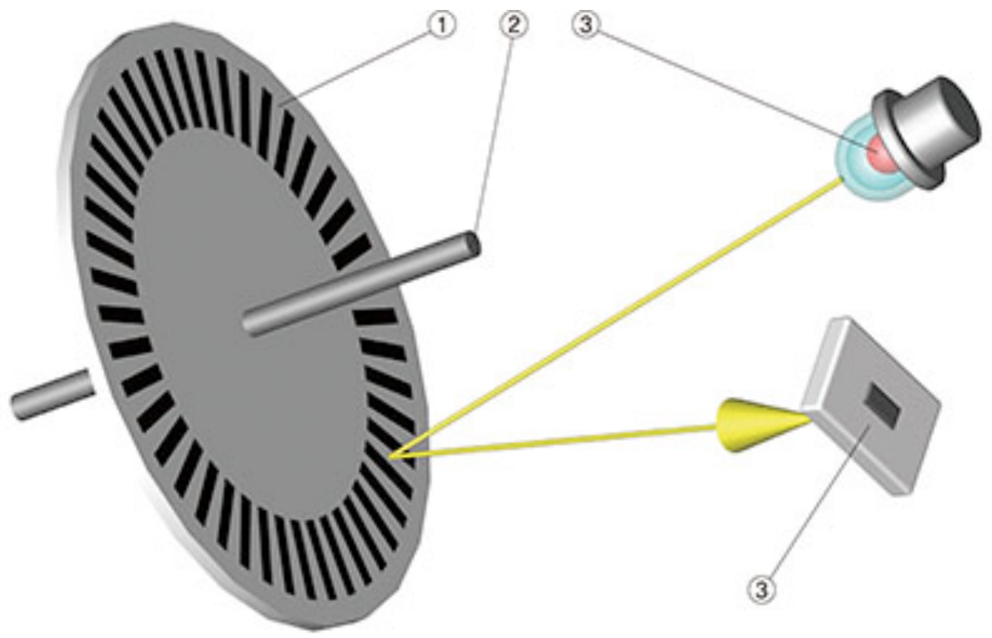
- actuators to produce motion
- proprioception to sense pose
- controller to regulate current to motor

Servo



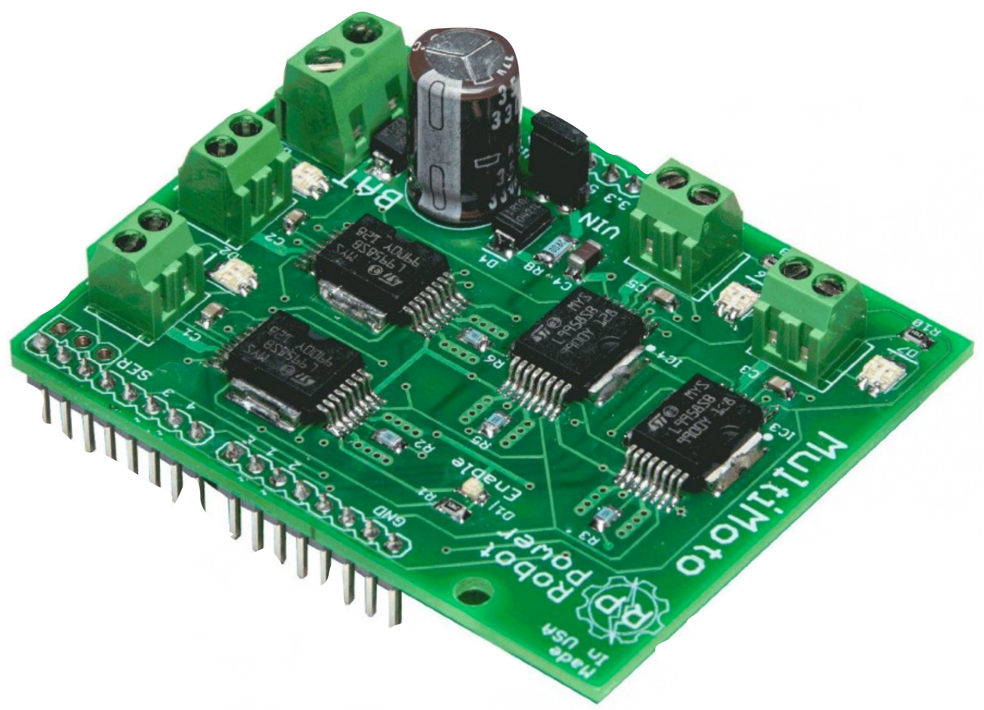
Actuator
(electric)





Proprioception
(optical encoder)

Controller
(4 channel H-Bridge)



Consider servos on
wheels of an inverted pendulum

ROB 550 Fall 2016 - https://youtu.be/9QUSexcsSZk

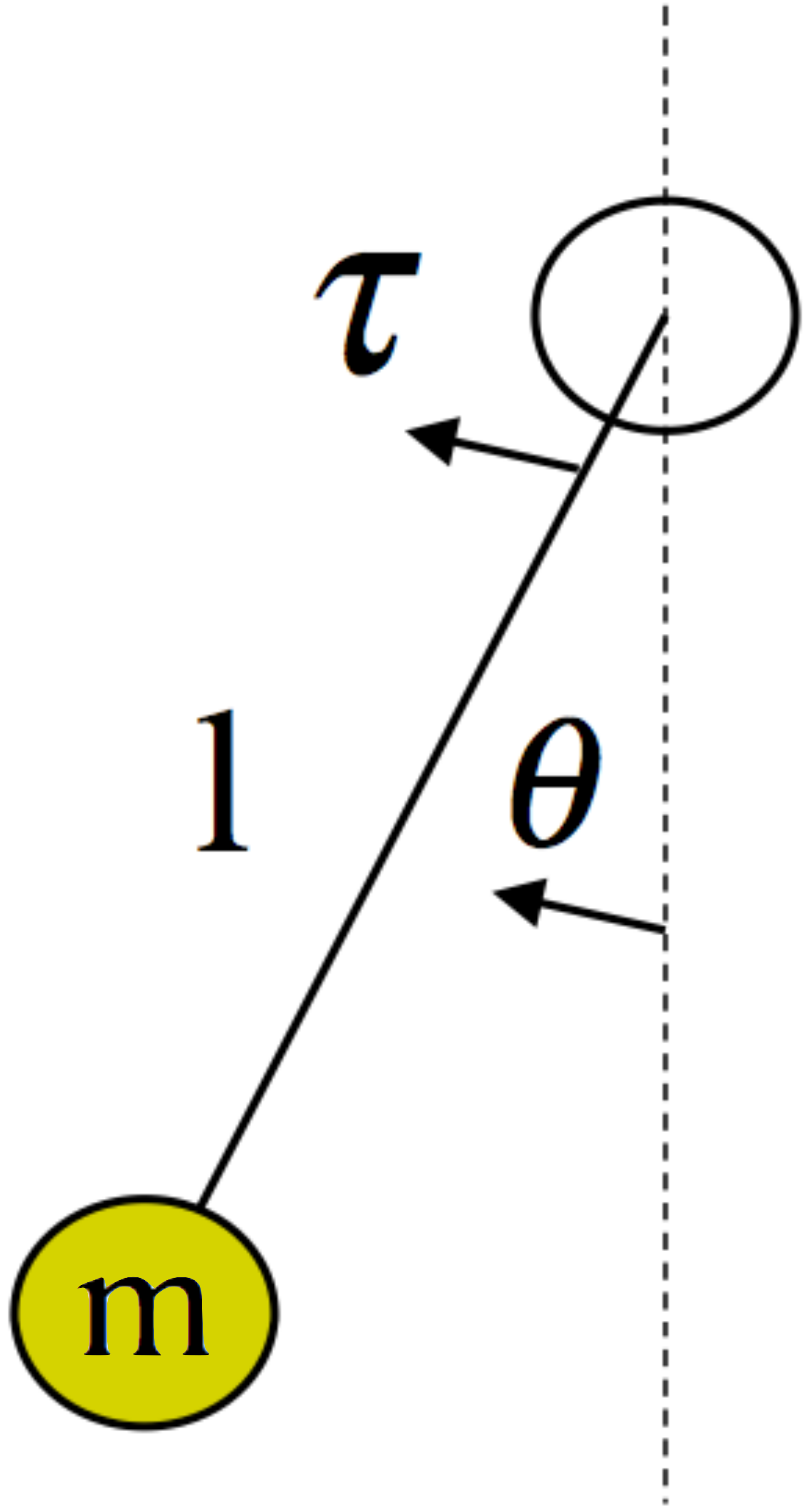

Design, build, and control inverted pendulum robot

What is an equation of motion?

Example: Pendulum

Equation of motion
(with rotational inertia I)

$$\underbrace{I}_{\text{Inertia}}\underbrace{\ddot{\theta}}_{\text{acceleration}} = \underbrace{-mgl\sin(\theta)+\tau}_{\text{net force}}$$



Motor produces torque (angular force)

Angle expresses pendulum range of motion

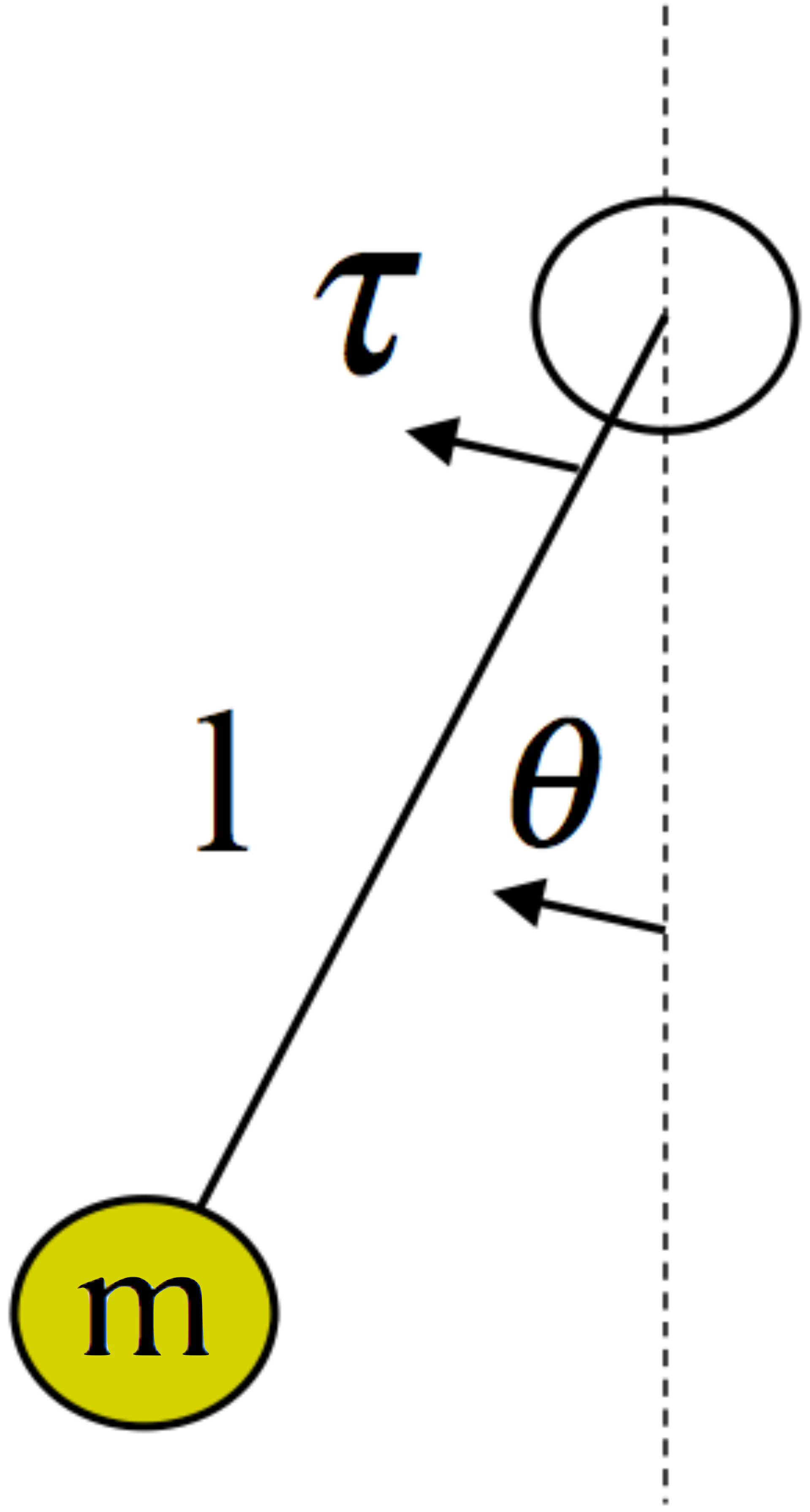
Pendulum of length *l* with point mass *m*

Example: Pendulum

Equation of motion
(with rotational inertia I)

$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$

angular acceleration
(second time derivative)



Motor produces torque
(angular force)

Angle expresses pendulum
range of motion

Pendulum of length *l* with
point mass *m*

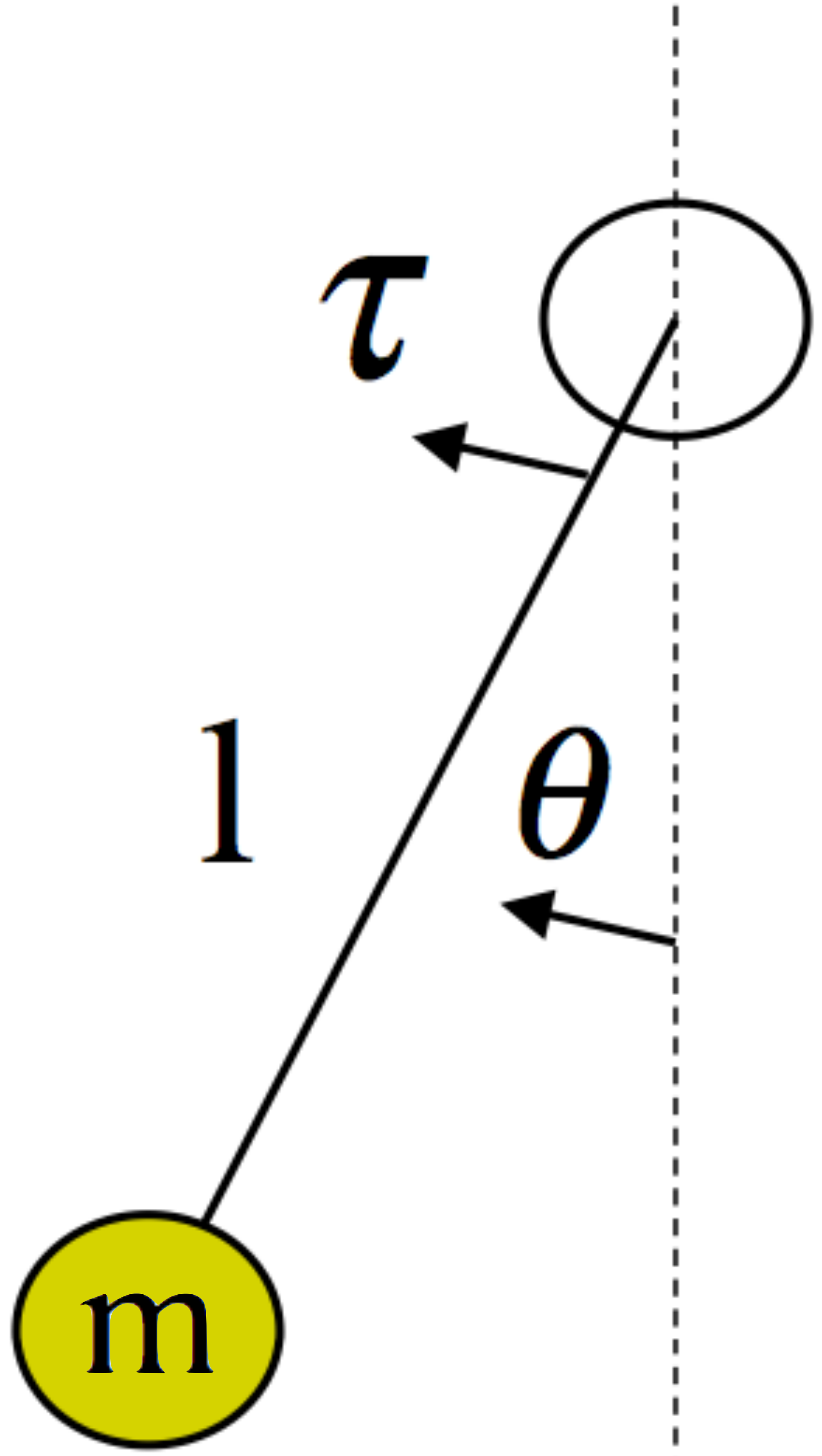
Example: Pendulum

Equation of motion
(with rotational inertia I)

$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$

angular acceleration
(second time derivative)

or $\dfrac{d\dot{\theta}}{dt}$ as the rate of change of the velocity of the pendulum angle



Motor produces torque (angular force)

Angle expresses pendulum range of motion

Pendulum of length l with point mass m

Example: Pendulum

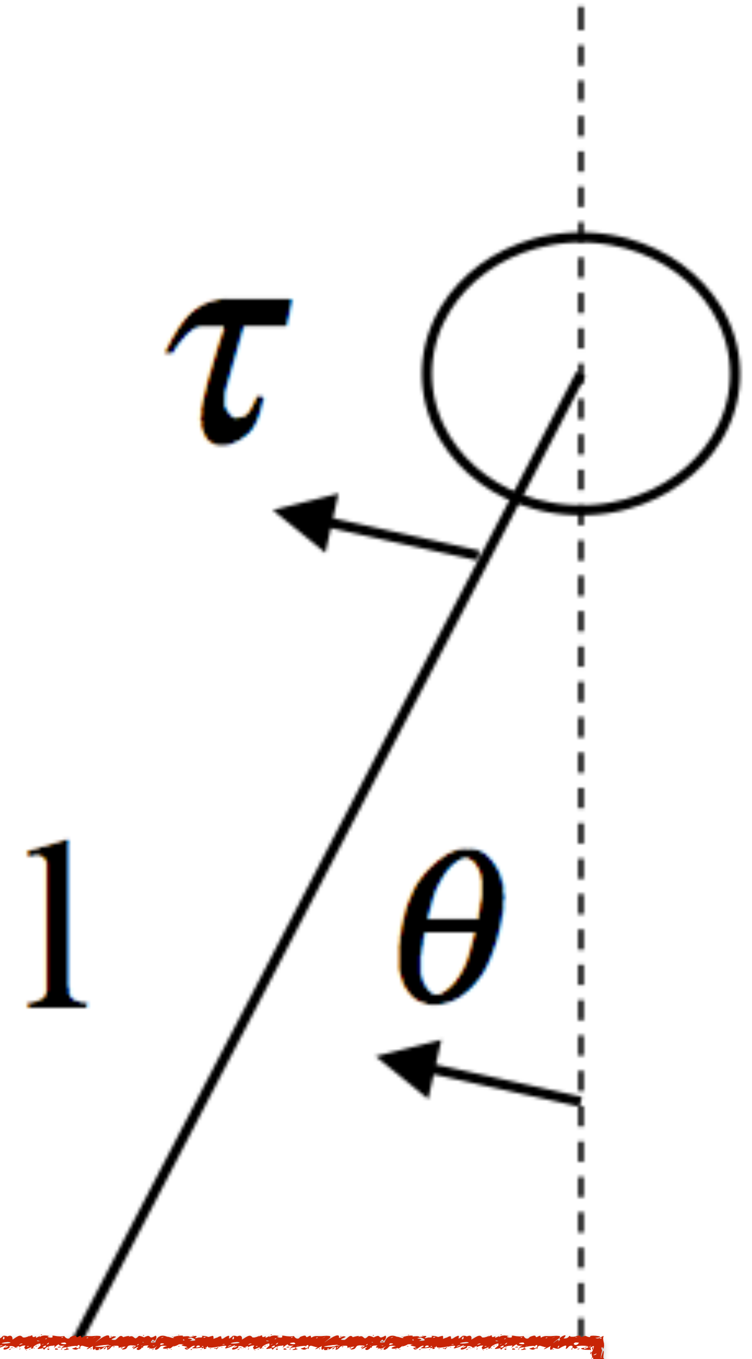
Equation of motion
(with rotational inertia I)

$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$

angular acceleration
(second time derivative)

or $\dfrac{d\dot{\theta}}{dt}$ as the rate of change of the velocity of the pendulum angle

or $\dfrac{d^2\theta}{dt^2}$ as the rate of change of the rate of change of the pendulum angle



Motor produces torque (angular force)

Angle expresses pendulum range of motion

Pendulum of length l with point mass m

$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$

How was the
equation of motion derived?

$$\frac{d^2\theta}{dt^2} \qquad \frac{d\dot{\theta}}{dt}$$

$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$

We are going to need some math

$$\frac{d^2\theta}{dt^2} \qquad \frac{d\dot{\theta}}{dt}$$

Derivative

From Wikipedia, the free encyclopedia

The **derivative** of a function of a real variable measures the sensitivity to change of a quantity (a function or dependent variable) which is determined by another quantity (the independent variable).

$$\frac{d\theta}{dt}$$

is the rate of change of the pendulum angle θ
with respect to the variable for time t

$$\theta = f(t)$$

assume that the pendulum angle
is a function over time

Derivative

From Wikipedia, the free encyclopedia

The **derivative** of a function of a real variable measures the sensitivity to change of a quantity (a function or dependent variable) which is determined by another quantity (the independent variable).

$$\frac{df}{da}$$

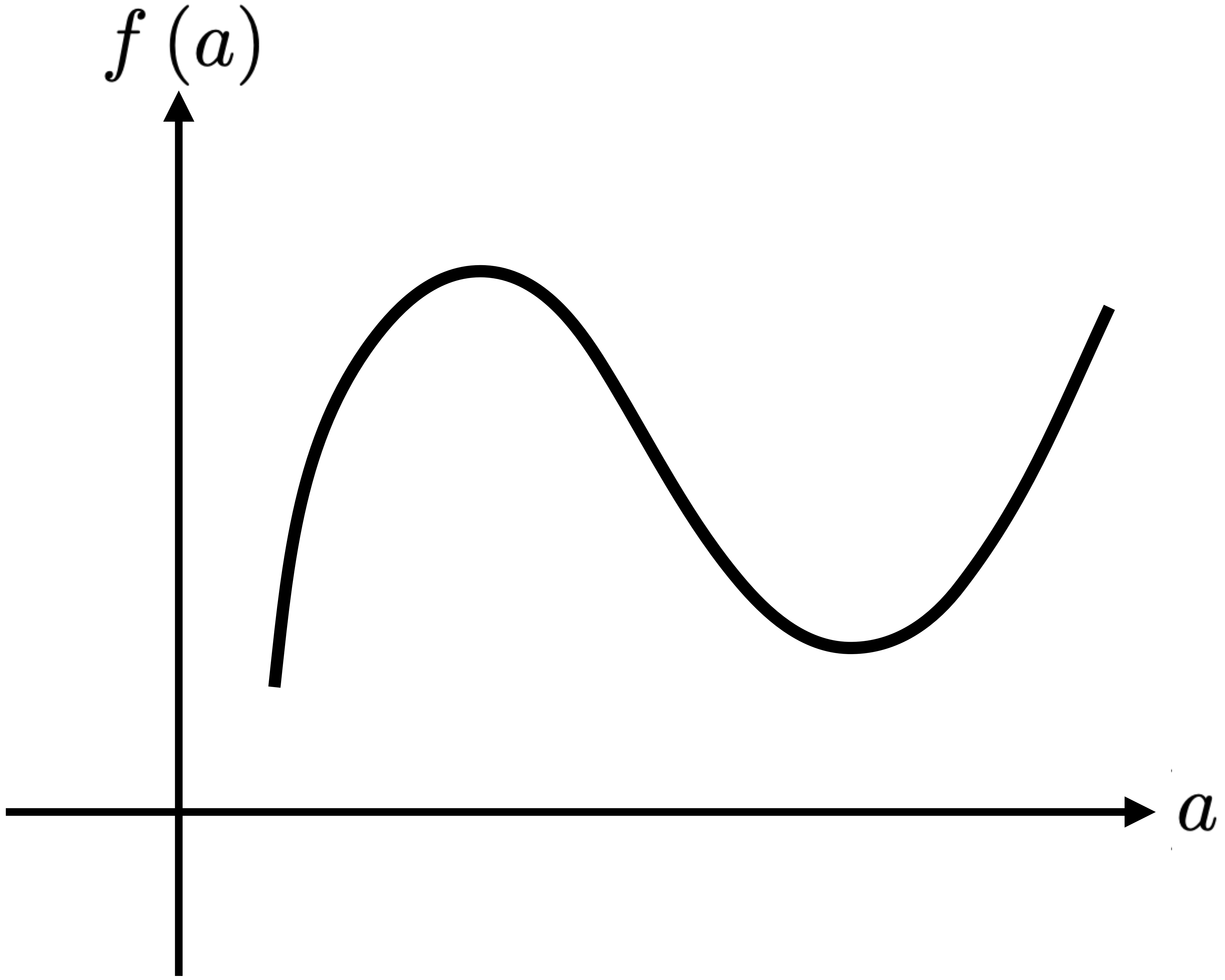
is the rate of change of some function $f(a)$ with respect to some variable a

Derivative

From Wikipedia, the free encyclopedia

The **derivative** of a function of a real variable measures the sensitivity to change of a quantity (a function or dependent variable) which is determined by another quantity (the independent variable).

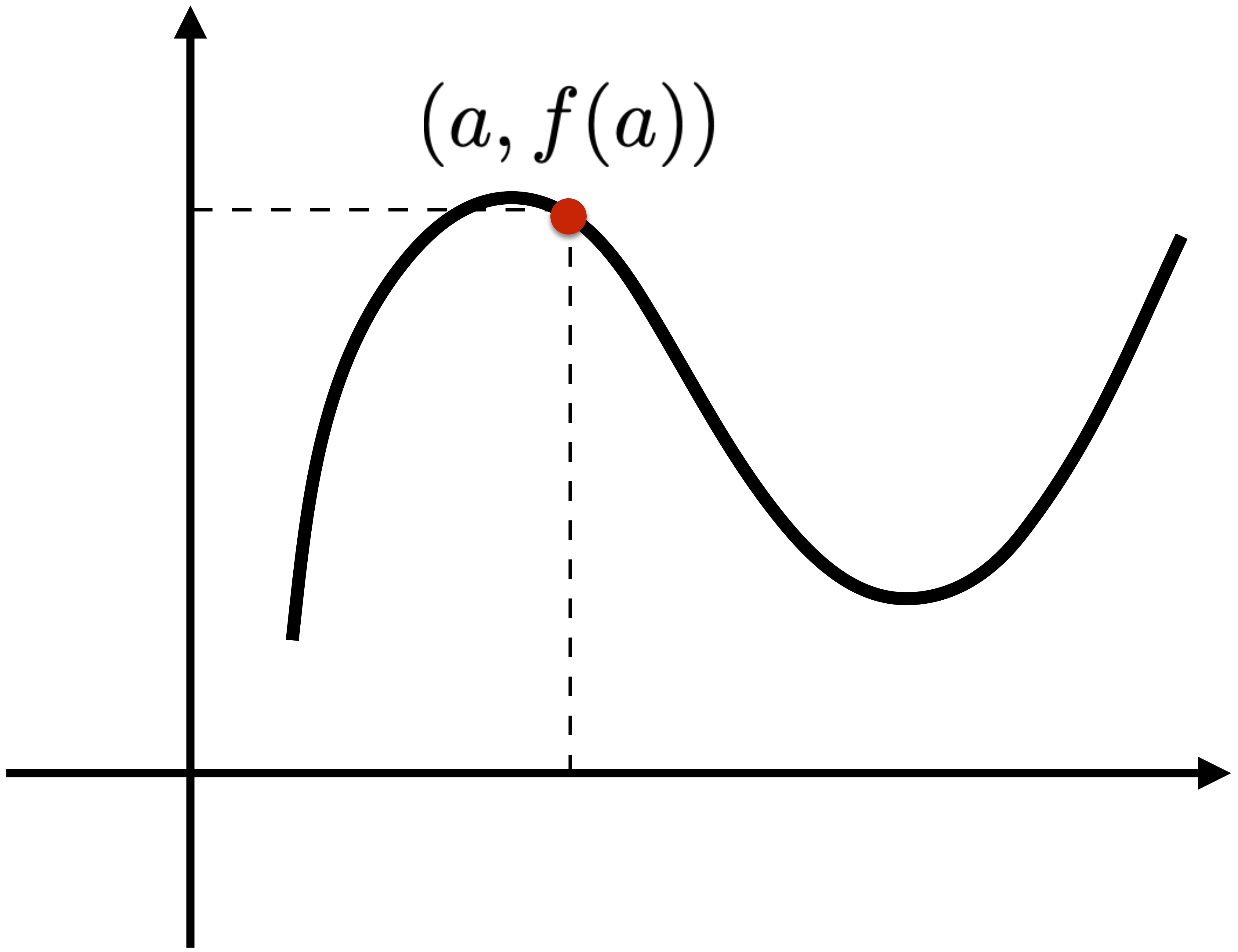
Consider a function $f\left(a\right)$ over variable a



Derivative

Consider a function $f(a)$ over variable a

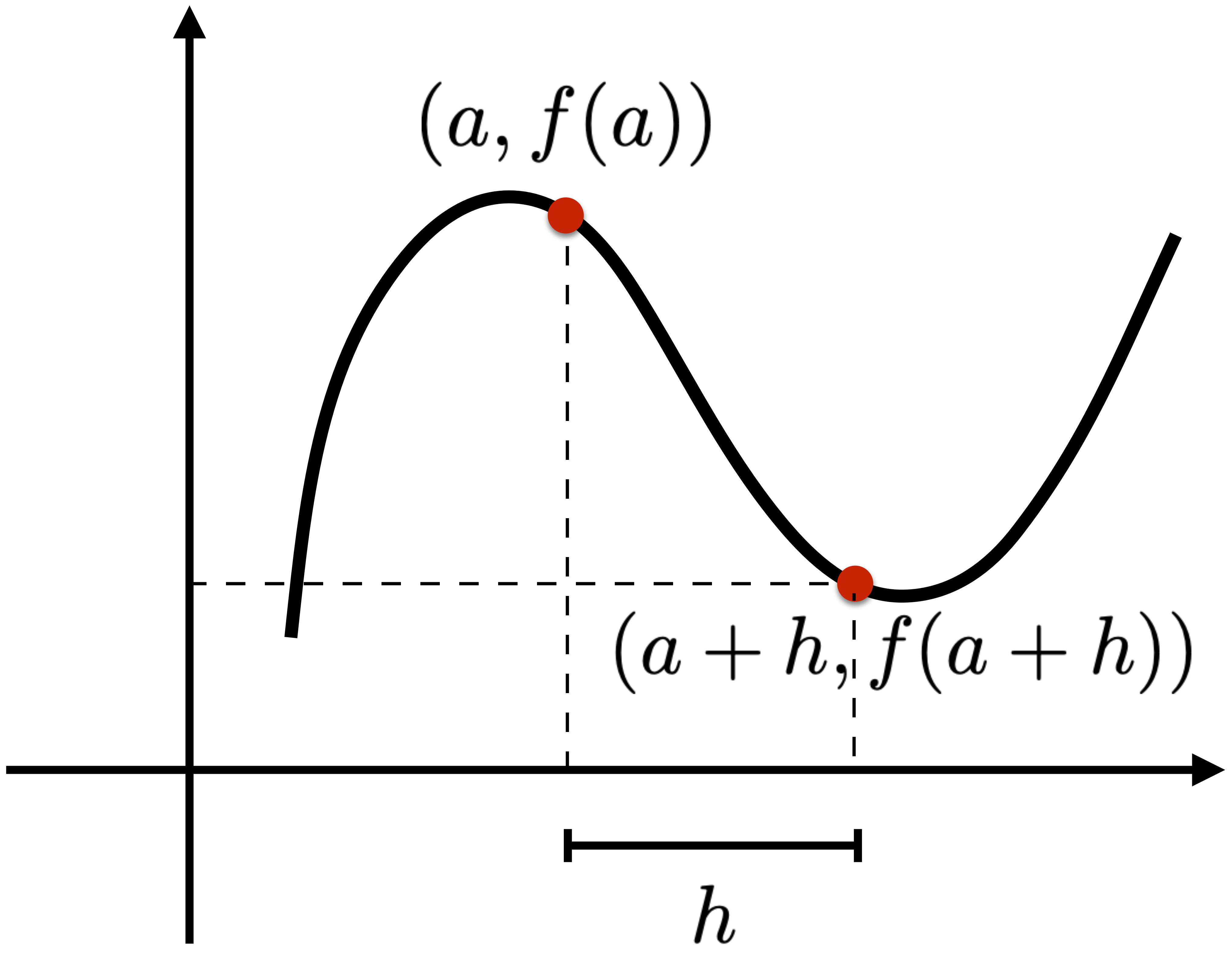
Evaluate $f(a)$ at a



Derivative

Consider a function $f(a)$ over variable a

Evaluate $f(a)$ at a and a nearby point $a + h$



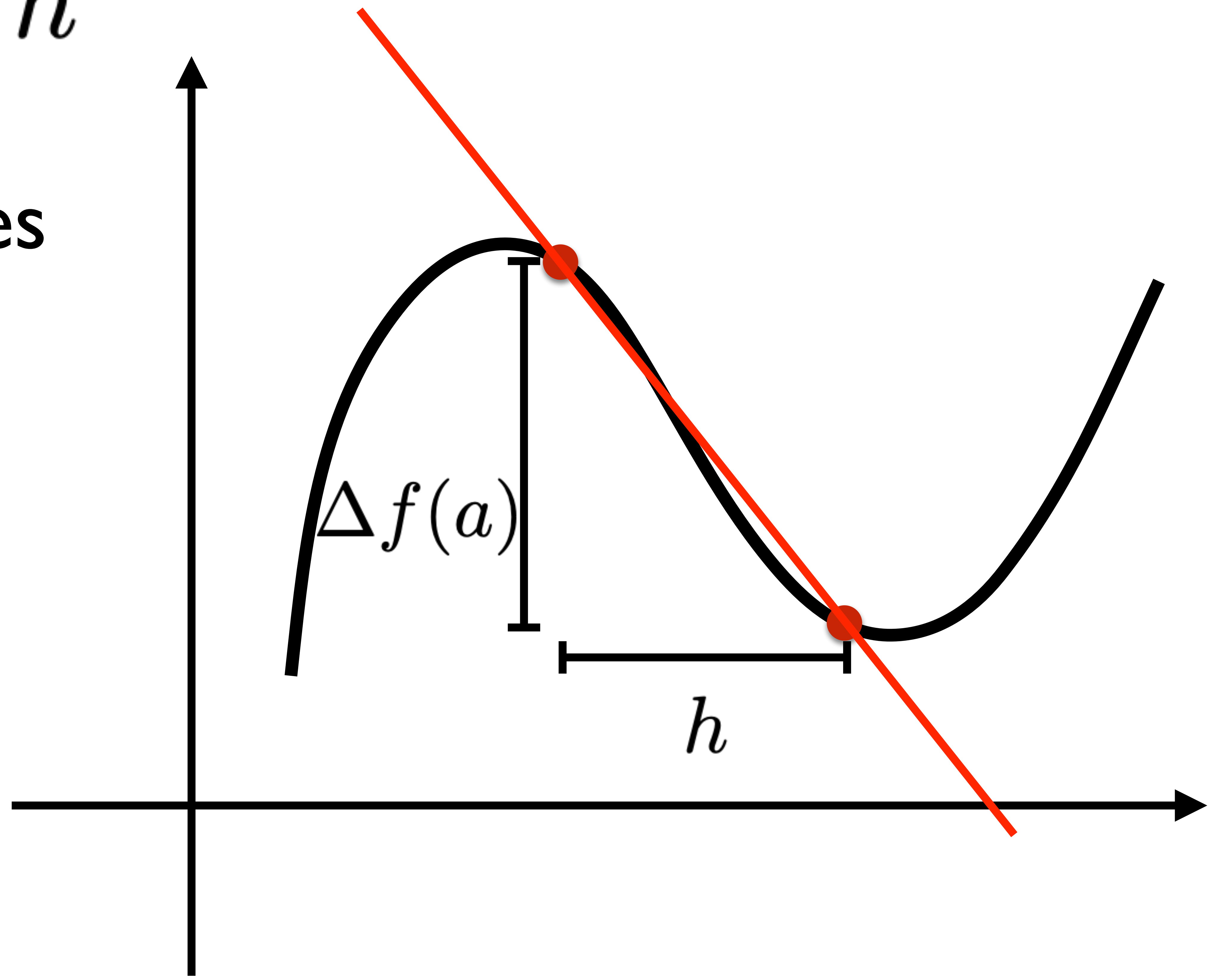
Derivative

Consider a function $f(a)$ over variable a

Evaluate $f(a)$ at a and a nearby point $a+h$

Slope of line between these points approximates the rate of change of the function

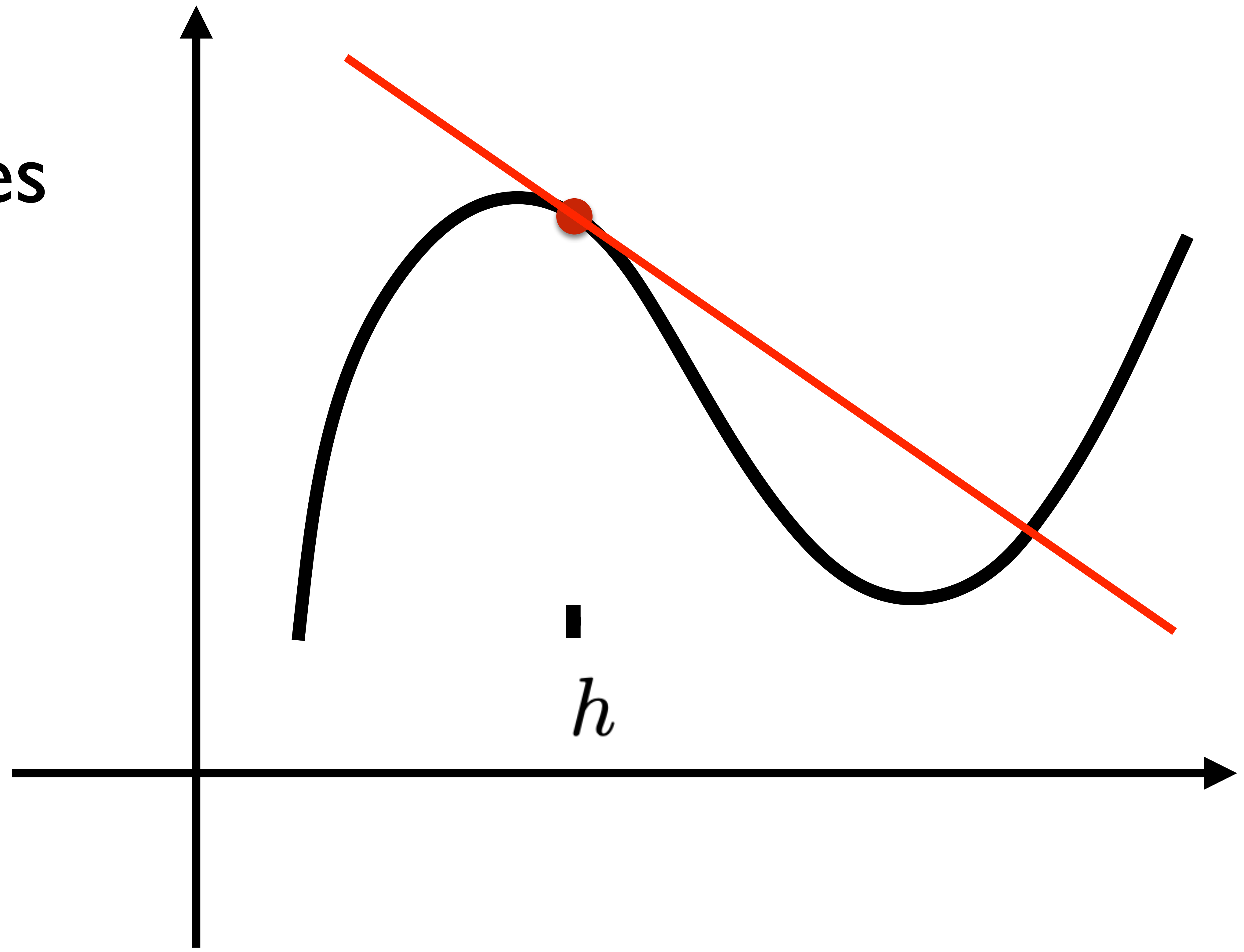
$$\frac{\Delta f(a)}{\Delta a} = \frac{f(a+h) - f(a)}{h}$$



Derivative

Consider a function $f\,(a)$ over variable a

Evaluate $f\,(a)$ at a and a nearby point $a+h$



Slope of line between these points approximates the rate of change of the function

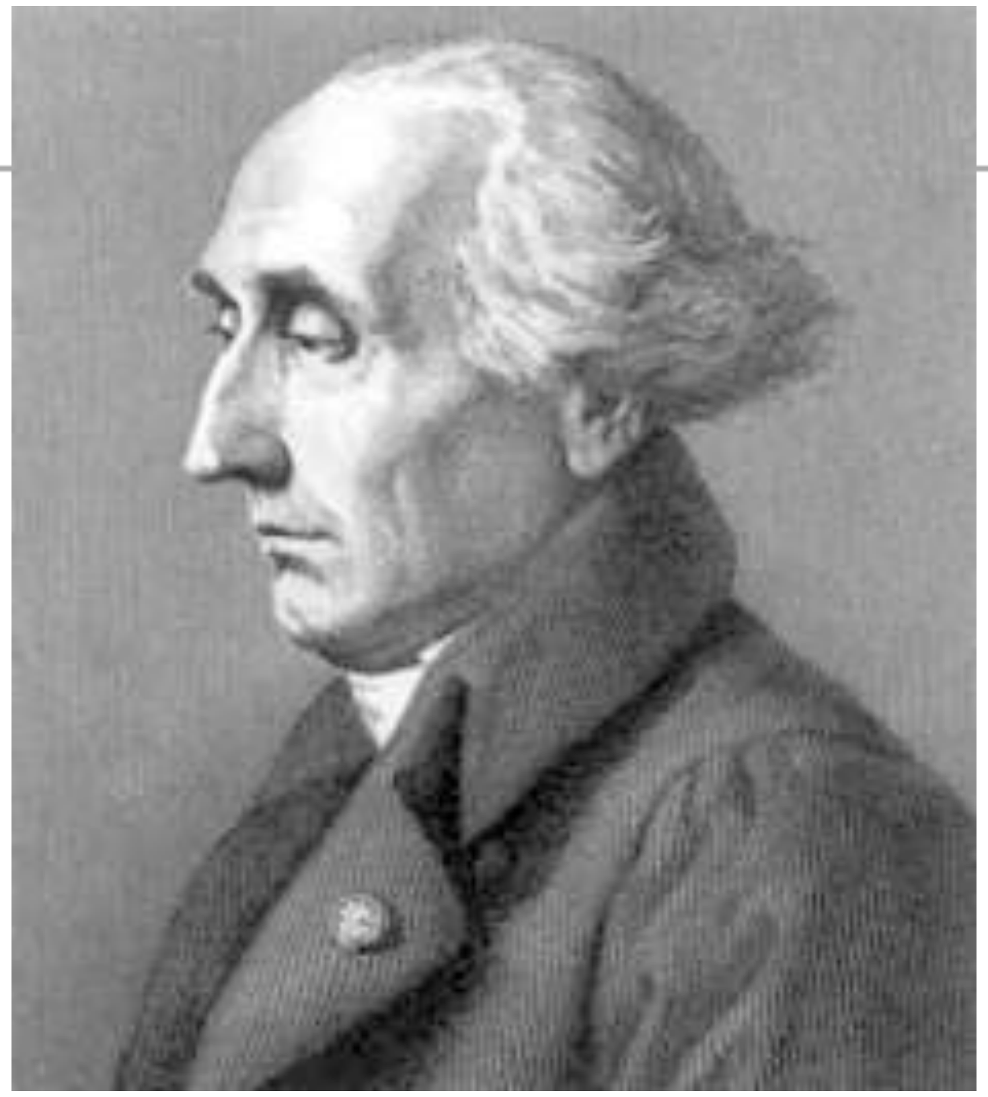
$$\frac{\Delta f(a)}{\Delta a} = \frac{f\,(a+h) - f\,(a)}{h}$$

Derivative is this slope as $a+h$ gets as close as possible to a

$$f\,'(a) = \lim_{h \to 0} \frac{f\,(a+h) - f\,(a)}{h}$$

Derivative

Gottfried Wilhelm Leibniz (1646-1716)



Joseph-Louis Lagrange (1736-1813)

Newton's Notation (assuming *a* is time)

Leibniz's Notation (assuming function *y = f(x)*)

Lagrange's Notation

$$\dot{y} = \frac{dy}{dx} = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

```
$\dot{y} = \frac{dy}{dx} = f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$
```

will produce this expression in LaTeX math mode

$$\dot{y} = \frac{dy}{dx} = f'(a)=\lim_{h\to 0}\frac{f\left(a+h\right)-f\left(a\right)}{h}$$

First derivative $\dot{y}$ or $\dfrac{dy}{dx}$ or f'

$$\frac{\Delta f(a)}{\Delta a} = \frac{f\left(a+h\right)-f\left(a\right)}{h}$$

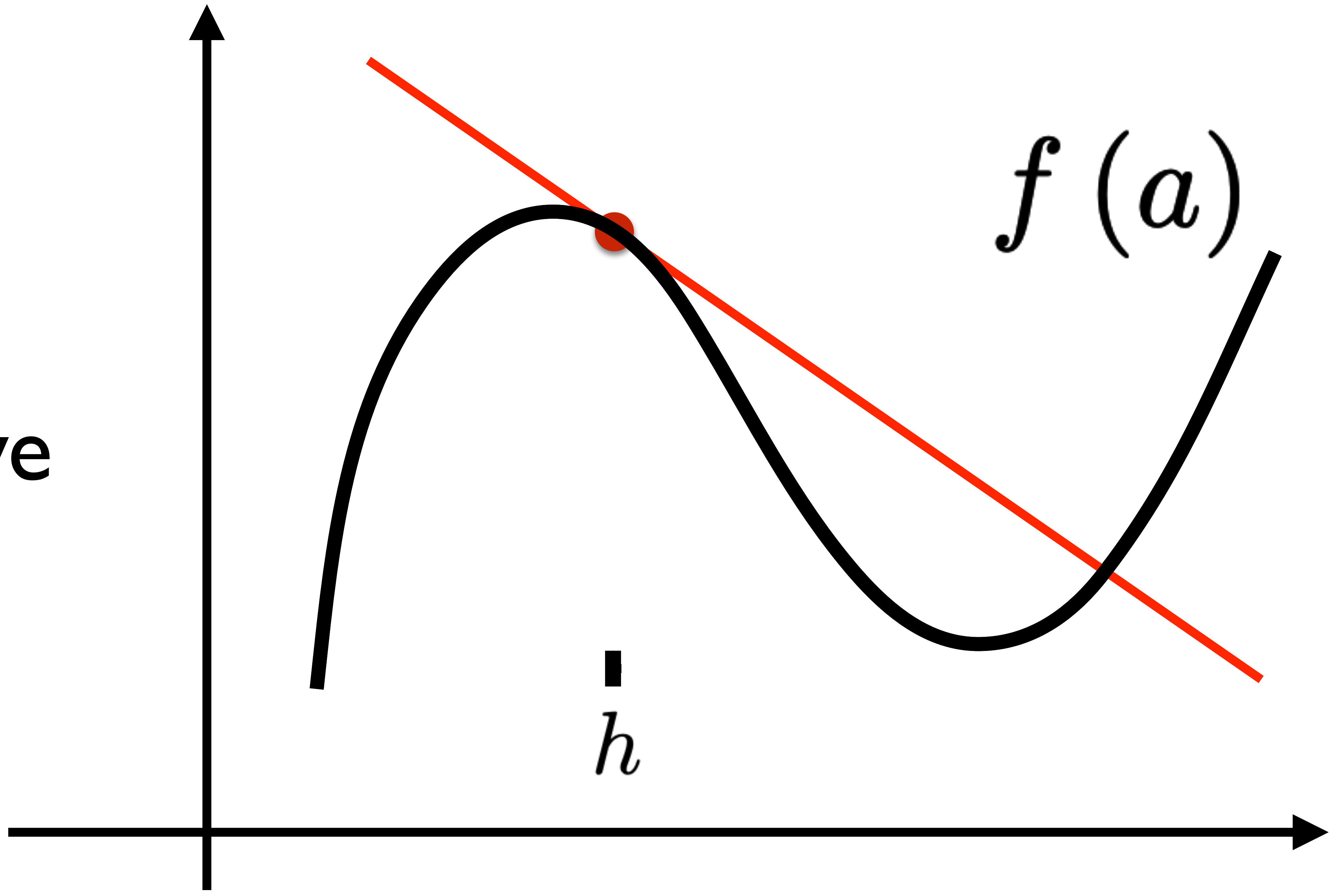
First derivative $\dot{y}$ or $\dfrac{dy}{dx}$ or f'

$$\frac{\Delta f(a)}{\Delta a} = \frac{f\left(a+h\right) - f\left(a\right)}{h}$$

Second derivative $\ddot{y}$ or $\dfrac{d^2y}{dx^2}$ or f''

$$\frac{\Delta^2 f(a)}{\Delta a^2} = \frac{\frac{f(a+h)-f(a)}{h} - \frac{f(a)-f(a-h)}{h}}{h}$$

Differentiation: the process of finding a derivative



For functions expressed in closed form, there are rules for differentiation such as:

Power rule

$$\frac{d}{dx}x^n = nx^{n-1},$$

$$n \neq 0.$$

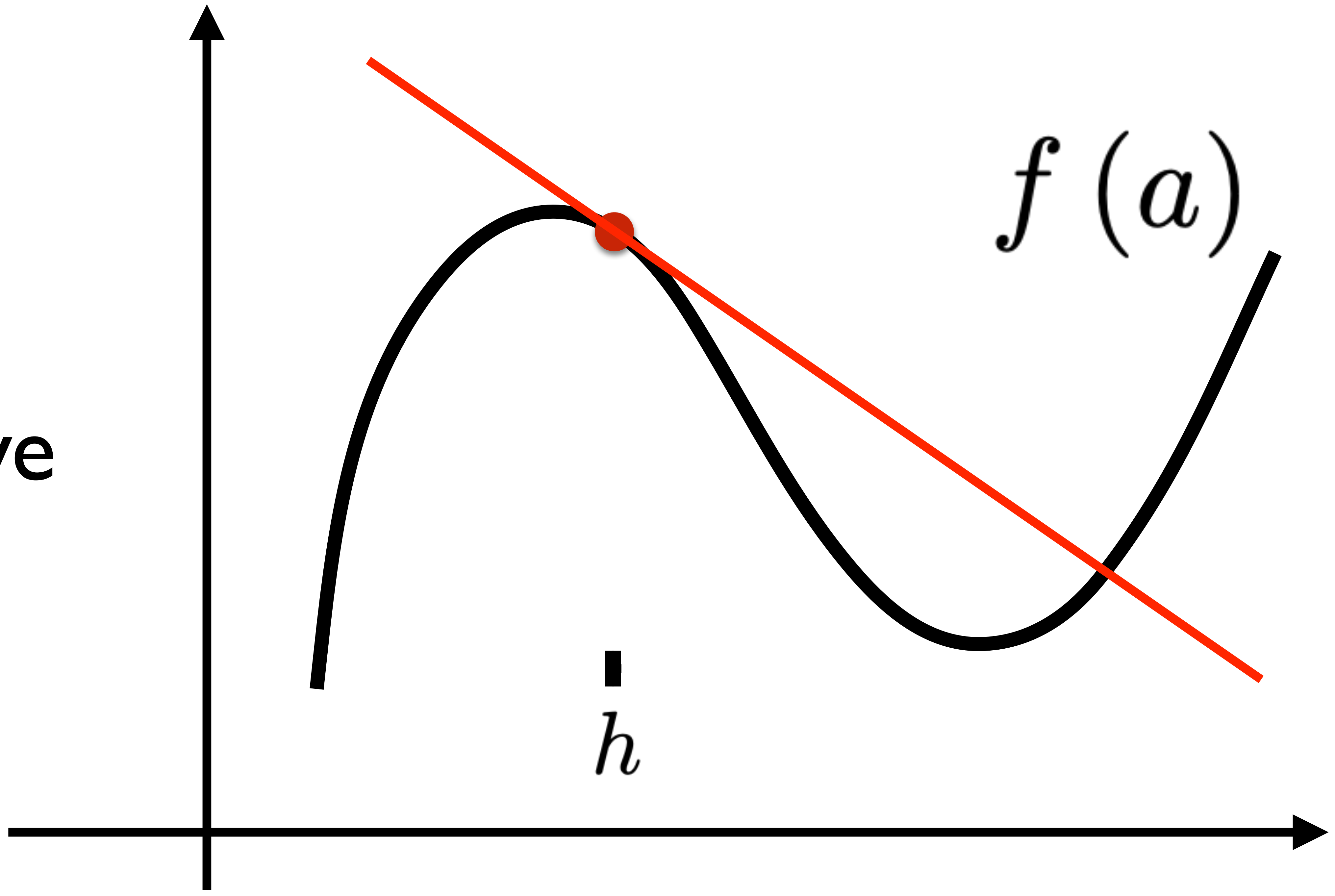
Product rule

$$\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

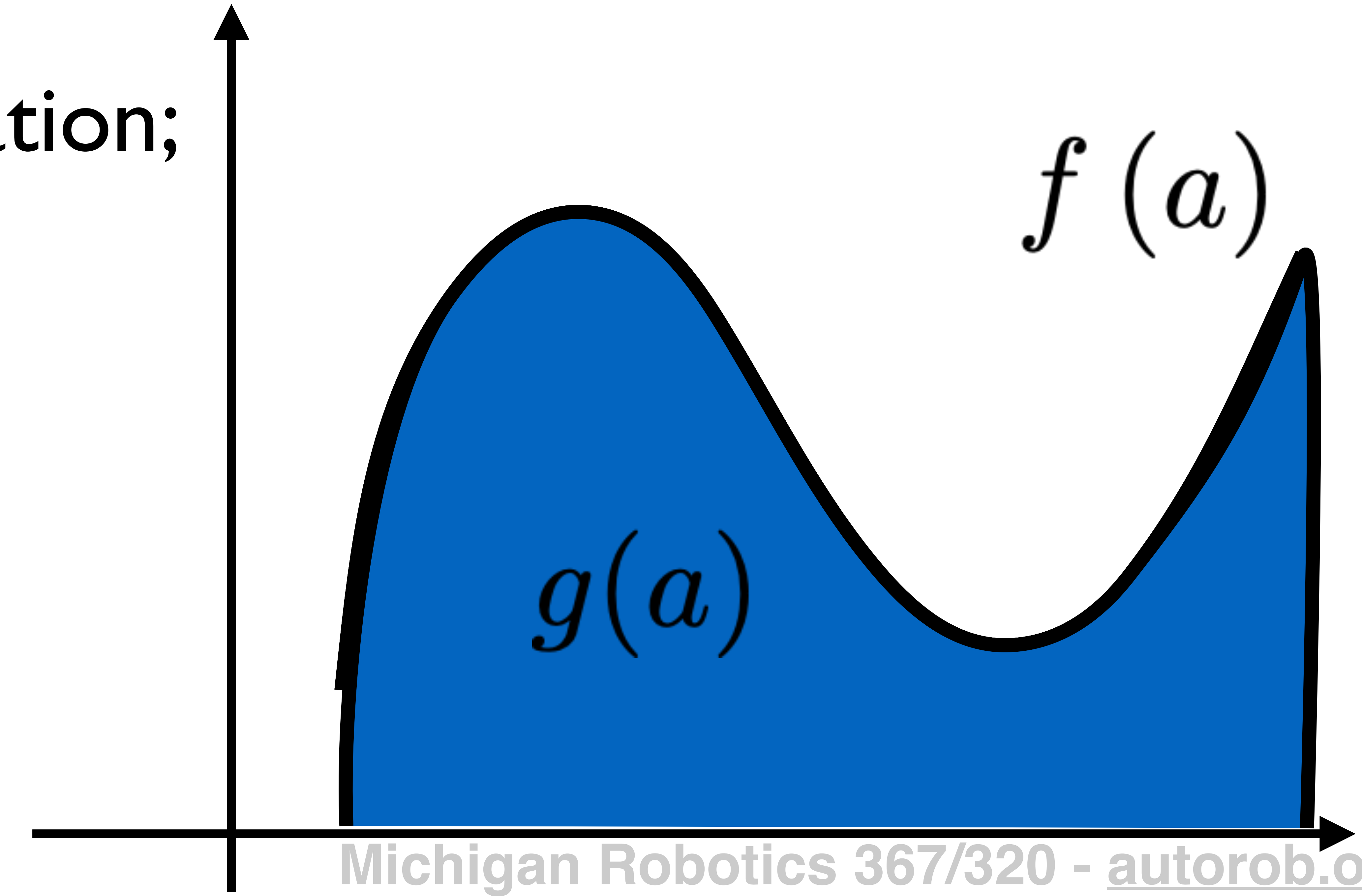
Chain rule

$$[f(g(x))]' = f'(g(x))g'(x)$$

Differentiation: the process of finding a derivative

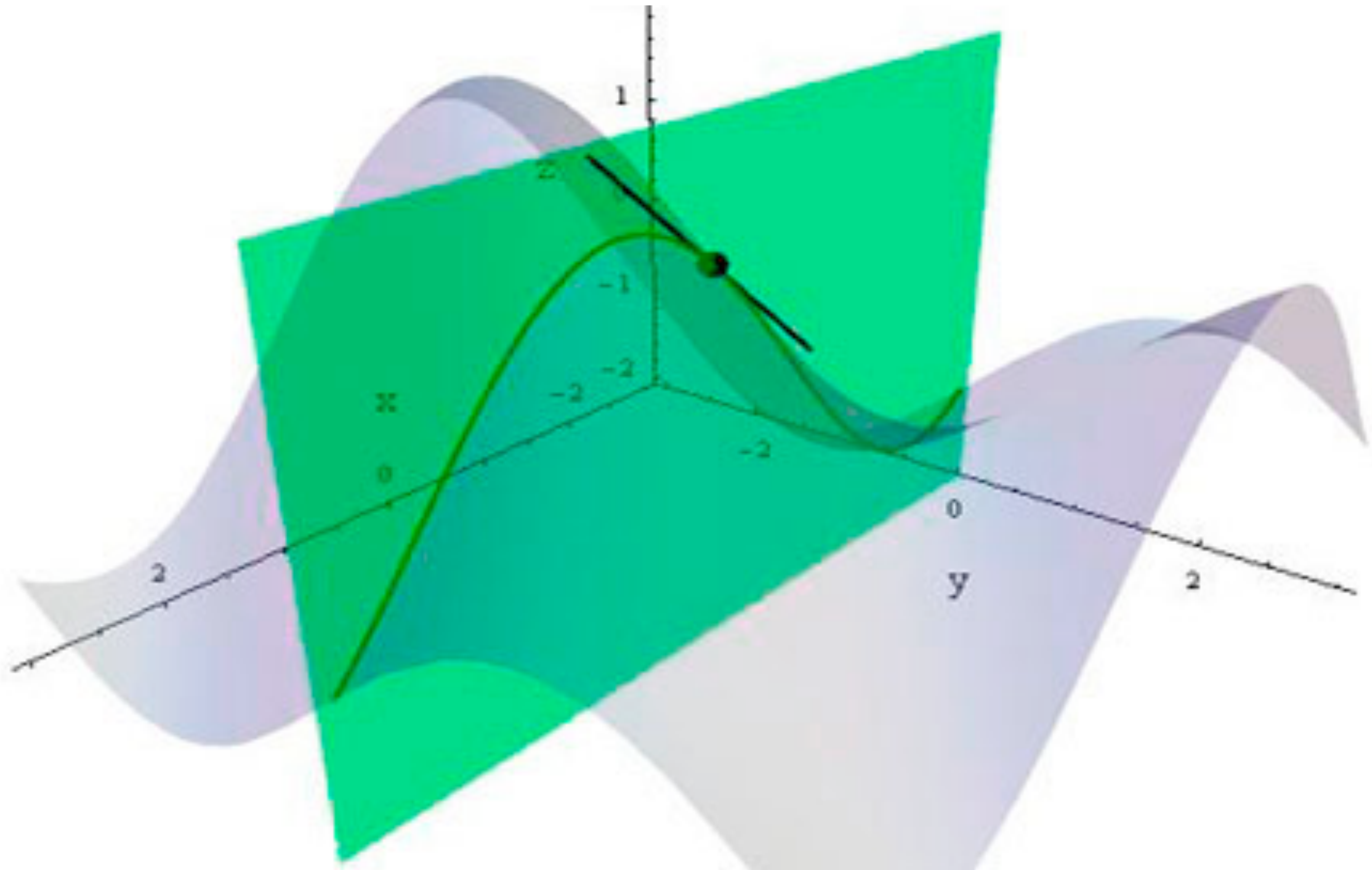


Integration is the inverse operation of differentiation; finding the area under the function



$$\frac{d}{dx}f(x) = g(x), \int g(x)\,dx = f(x) + C$$

Differential equations addresses differentiation for functions of multiple variables



Consider $z = f(x, y, t)$

and $x = x(t)\,, y = y(t)$

https://math.stackexchange.com/questions/607942/what-is-the-best-way-to-think-about-partial-derivatives

---

Partial derivative of z with respect to t is

$$\frac{\partial f}{\partial t}$$

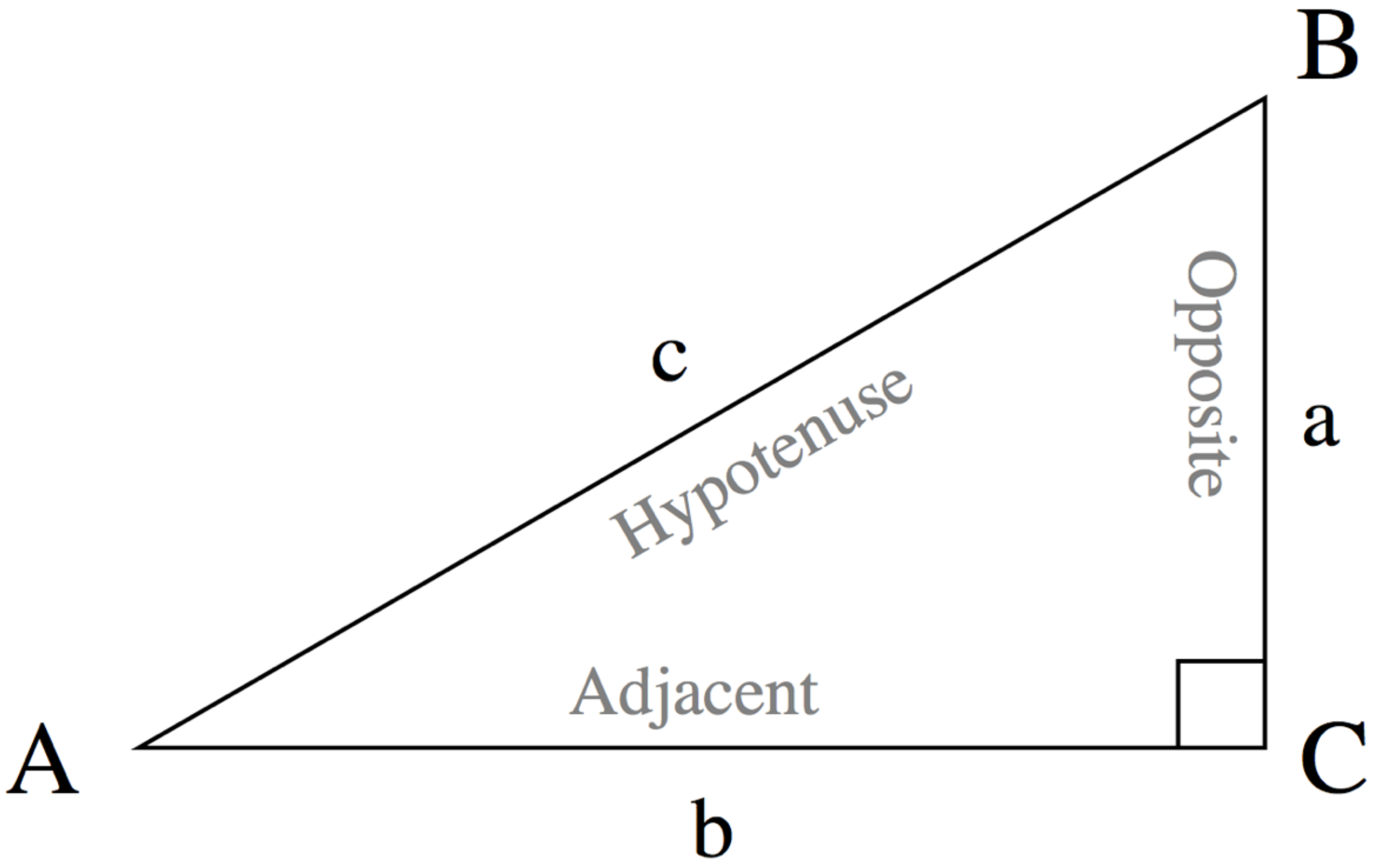
x and y treated as constants

Total derivative of z with respect to t is

$$\frac{\mathrm{d}\,f}{\mathrm{d}\,t} = \frac{\partial f}{\partial t}\frac{\mathrm{d}\,t}{\mathrm{d}\,t} + \frac{\partial f}{\partial x}\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}\,y}{\mathrm{d}\,t}$$

via the chain rule

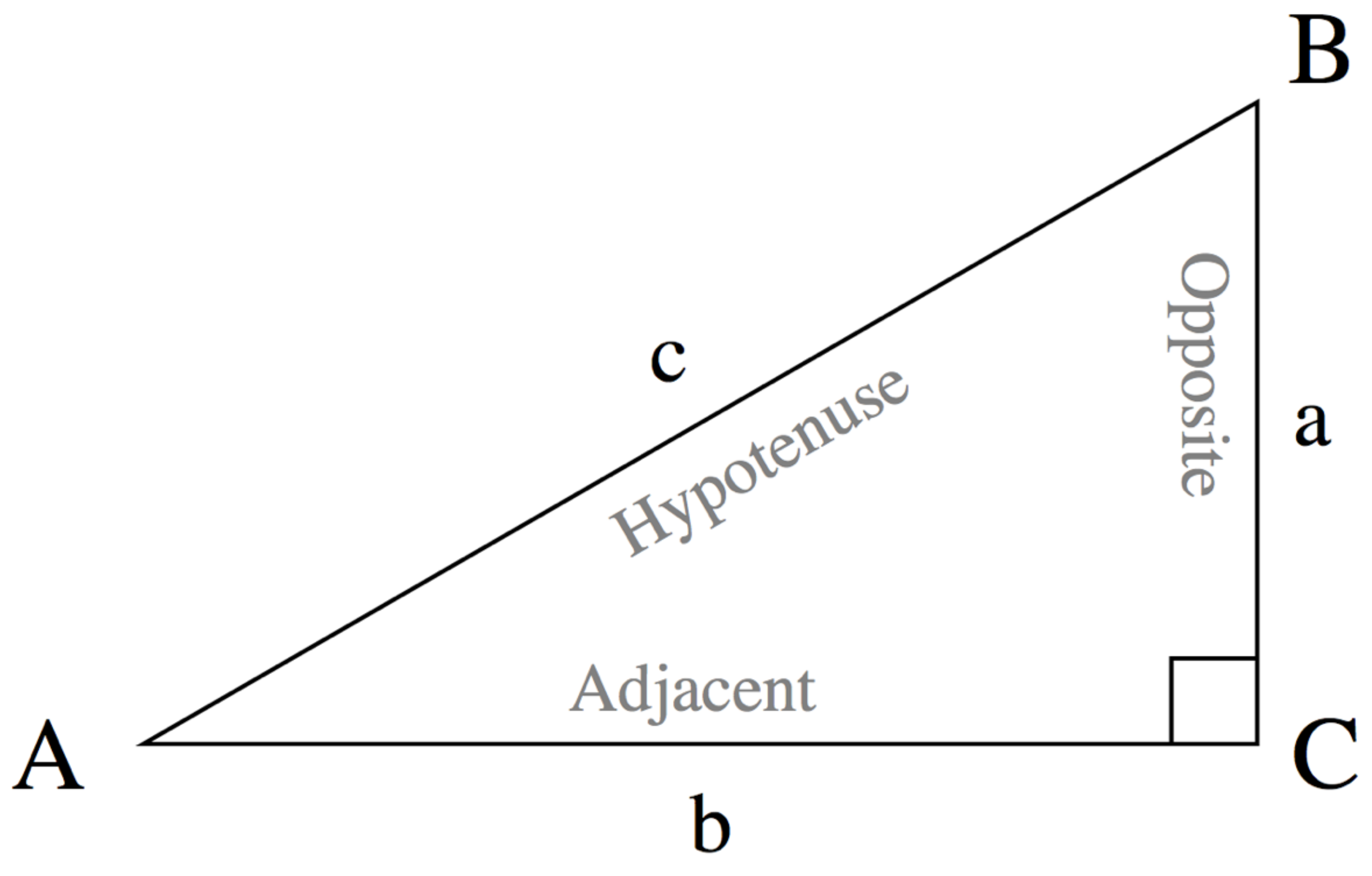
Don’t forget trig



Trigonometry

From Wikipedia, the free encyclopedia

Trigonometry (from Greek *trigōnon*, "triangle" and *metron*, "measure"[1]) is a branch of mathematics that studies relationships involving lengths and angles of triangles. The field emerged during the 3rd century BC from applications of geometry to astronomical studies.[2]




```
Math.sin(A)
```

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}.$$

```
Math.cos(A)
```

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}.$$

```
Math.tan(A)
```

$$\tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b} = \frac{\sin A}{\cos A}$$

```
Math.atan2(a,b)
```

$$\operatorname{atan2}(y,x) = \begin{cases} \arctan\frac{y}{x} & x > 0 \\ \arctan\frac{y}{x} + \pi & y \geq 0, x < 0 \\ \arctan\frac{y}{x} - \pi & y < 0, x < 0 \\ +\frac{\pi}{2} & y > 0, x = 0 \\ -\frac{\pi}{2} & y < 0, x = 0 \\ \text{undefined} & y = 0, x = 0 \end{cases}$$

Coming back …

$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$

How was the
equation of motion derived?

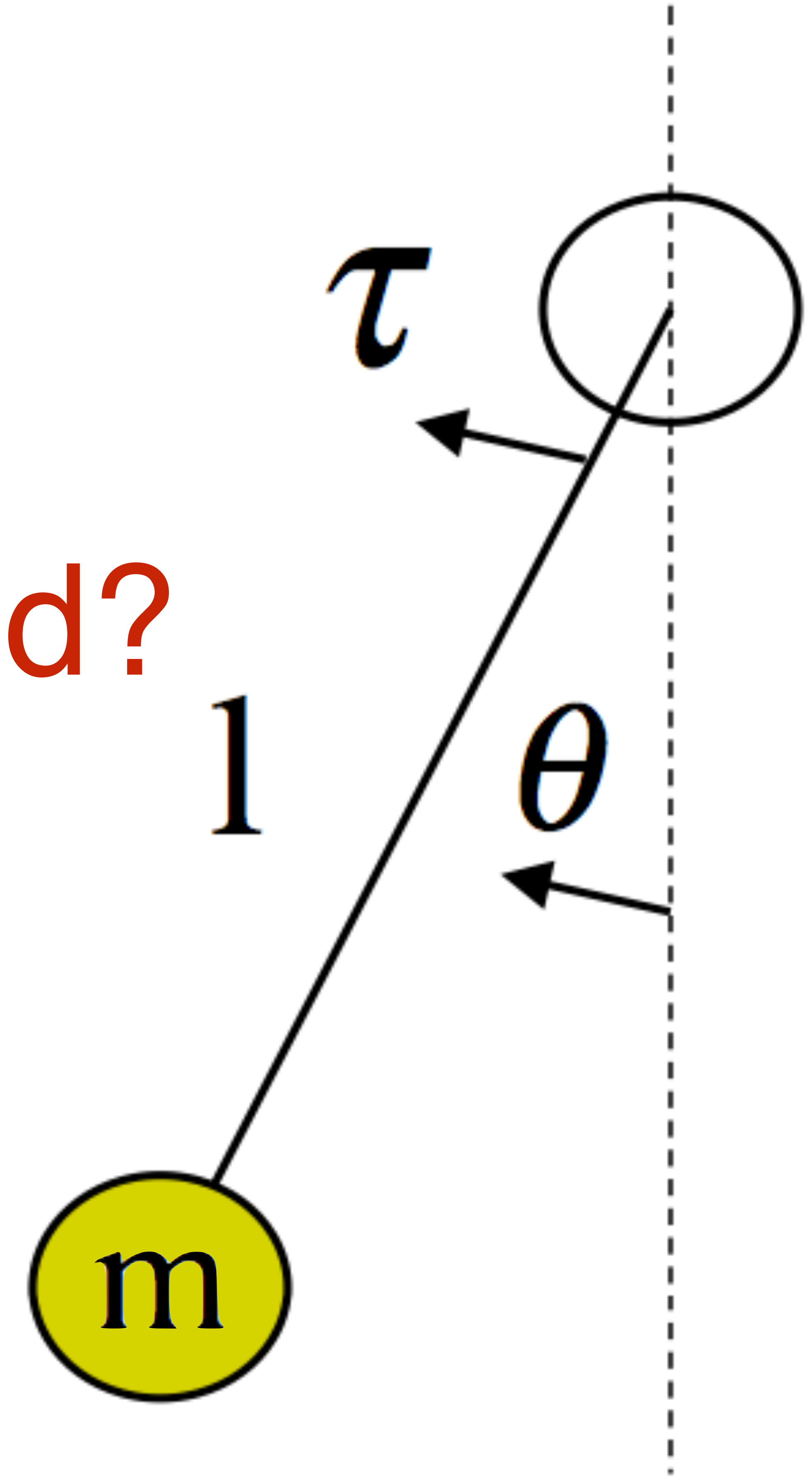
$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$

Example: Pendulum

Equation of motion
(with rotational inertia I)

$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$

How was this derived?



Motor produces torque (angular force)

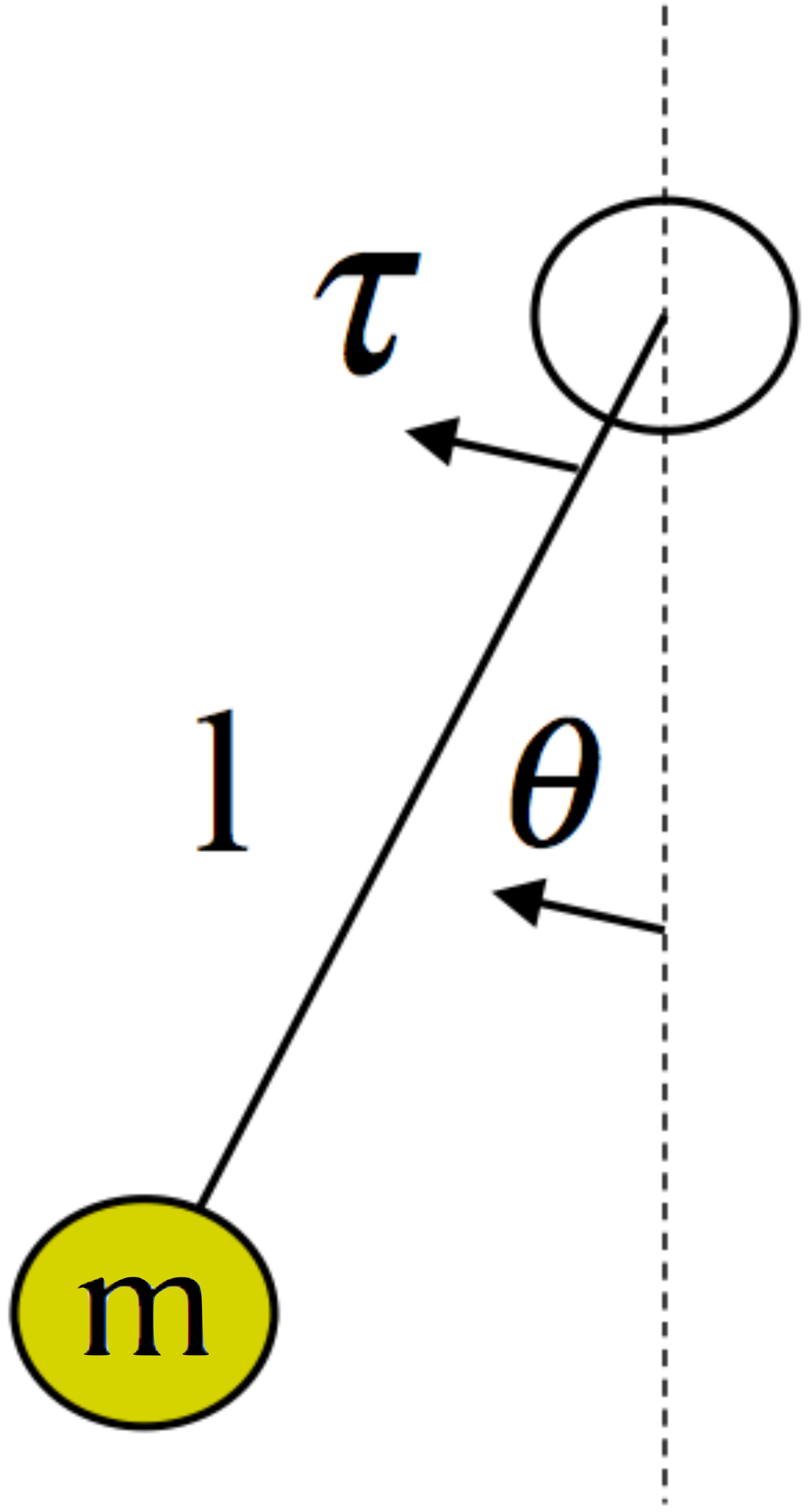
Angle expresses pendulum range of motion

Pendulum of length *l* with point mass *m*

Lagrangian Dynamics

Equation of motion
(with rotational inertia I)

$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$



Lagrangian is kinetic energy minus potential energy

$$L = T - U$$

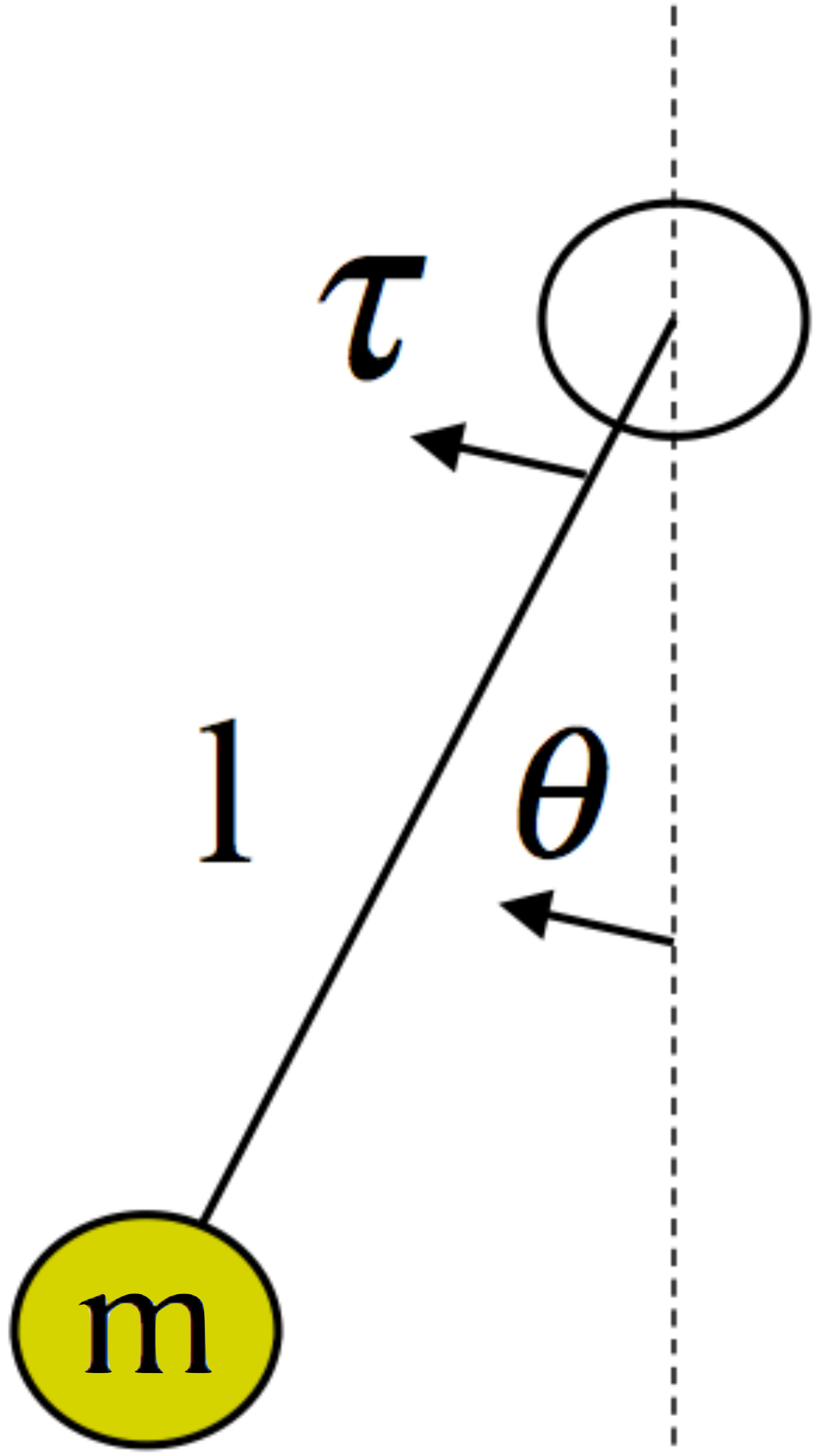
and used to generate equation of motion as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \tau_i$$

Lagrangian Dynamics

Equation of motion
(with rotational inertia I)

$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$



Lagrangian is kinetic energy minus potential energy

$$L = T - U$$

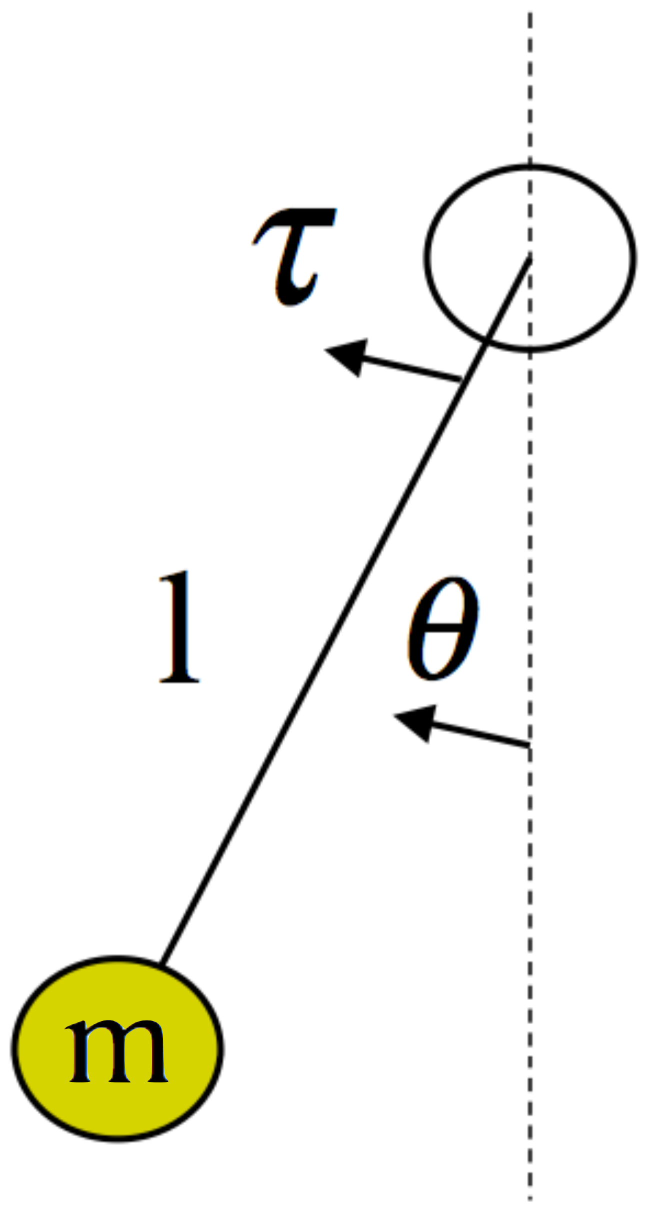
and used to generate equation of motion as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \tau_i$$

$\frac{\partial L}{\partial \theta_i}$ is the rate of change of the Lagrangian with respect to only the i^{th} degree of freedom

Equation of motion
(with rotational inertia I)

$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$



Lagrangian Dynamics for Pendulum

- Kinetic Energy $T = \frac{1}{2} I\dot{\theta}^2$

  $(1/2)mv^2$

- Potential Energy $U = mgl(1-\cos\theta)$

  mgh

- Lagrangian $\frac{1}{2} I\dot{\theta}^2 - mgl(1-\cos\theta)$

  $L = T - U$

- Equation of motion $I\ddot{\theta} = -mgl\sin(\theta) + \tau$

  $$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \tau_i$$

- Equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \boxed{\frac{\partial L}{\partial \theta_i}} = \tau_i$$

$$L = \frac{1}{2} I\dot{\theta}^2 - mgl(1-\cos\theta)$$

- Equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \tau_i$$

$$L = \frac{1}{2} I\dot{\theta}^2 - mgl(1-\cos\theta)$$

- Partial derivative of Lagrangian wrt. pendulum angle

$$\frac{\partial L}{\partial \theta_i} = \frac{\partial}{\partial \theta_i} \frac{1}{2} I\dot{\theta}^2 - mgl(1-\cos\theta)$$

- Equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \tau_i$$

$$L = \frac{1}{2} I\dot{\theta}^2 - mgl(1-\cos\theta)$$

- Partial derivative of Lagrangian wrt. pendulum angle

$$\frac{\partial L}{\partial \theta_i} = \frac{\partial}{\partial \theta_i} \overset{0}{\cancel{\frac{1}{2} I\dot{\theta}^2}} - mgl(1-\cos\theta)$$

$\frac{1}{2} I\dot{\theta}^2$ First term is constant wrt. θ

differentiates to zero

- Equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \tau_i$$

$$\boxed{L = \frac{1}{2}I\dot{\theta}^2 - mgl(1-\cos\theta)}$$

- Partial derivative of Lagrangian wrt. pendulum angle

$$\frac{\partial L}{\partial \theta_i} = \frac{\partial}{\partial \theta_i}\overset{0}{\cancel{\frac{1}{2}I\dot{\theta}^2}} - mgl(1-\cos\theta)$$

$\frac{1}{2}I\dot{\theta}^2$ First term is constant wrt. θ

differentiates to zero

Second term: $-mgl(1-\cos\theta)$

distribute multiplication: $-mgl + mgl\cos(\theta)$

cosine identity for differentiation: $\frac{d}{dx}\cos(x) = -\sin(x)$

yields: $-mgl\sin(\theta)$

- Equation of motion

$$\frac{d}{dt}\boxed{\frac{\partial L}{\partial \dot{\theta}_i}} - \frac{\partial L}{\partial \theta_i} = \tau_i$$

$$L = \frac{1}{2} I\dot{\theta}^2 - mgl(1-\cos\theta)$$

- Partial derivative of Lagrangian wrt. pendulum angle

$$\frac{\partial L}{\partial \theta_i} = -mgl\sin(\theta)$$

- Equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \tau_i$$

$$L = \frac{1}{2} I\dot{\theta}^2 - mgl(1-\cos\theta)$$

- Partial derivative of Lagrangian wrt. pendulum angle

$$\frac{\partial L}{\partial \theta_i} = -mgl\sin(\theta)$$

- Partial derivative of Lagrangian wrt. pendulum velocity

$$\frac{\partial L}{\partial \dot{\theta}_i} = \frac{\partial}{\partial \dot{\theta}_i} \frac{1}{2} I\dot{\theta}^2 - mgl(1-\cos\theta)$$

- Equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \tau_i$$

$$\boxed{L = \frac{1}{2} I\dot{\theta}^2 - mgl(1-\cos\theta)}$$

- Partial derivative of Lagrangian wrt. pendulum angle

$$\frac{\partial L}{\partial \theta_i} = -mgl\sin(\theta)$$

- Partial derivative of Lagrangian wrt. pendulum velocity

$$\frac{\partial L}{\partial \dot{\theta}_i} = \frac{\partial}{\partial \dot{\theta}_i} \frac{1}{2} I\dot{\theta}^2 - \overset{0}{\cancel{mgl(1-\cos\theta)}}$$

Second term is constant wrt. $\dot{\theta}$

- Equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \tau_i$$

$$\boxed{L = \frac{1}{2} I\dot{\theta}^2 - mgl(1-\cos\theta)}$$

- Partial derivative of Lagrangian wrt. pendulum angle

$$\frac{\partial L}{\partial \theta_i} = -mgl\sin(\theta)$$

- Partial derivative of Lagrangian wrt. pendulum velocity

$$\frac{\partial L}{\partial \dot{\theta}_i} = \frac{\partial}{\partial \dot{\theta}_i} \frac{1}{2} I\dot{\theta}^2 - \overset{0}{\cancel{mgl(1-\cos\theta)}}$$

Second term is constant wrt. $\dot{\theta}$

Apply power rule to first term: $f'(x) = rx^{r-1}$

differentiates to: $I\dot{\theta} = \frac{\partial}{\partial \dot{\theta}_i} \frac{1}{2} I\dot{\theta}^2$

- Equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \tau_i$$

$$L = \frac{1}{2} I\dot{\theta}^2 - mgl(1-\cos\theta)$$

- Partial derivative of Lagrangian wrt. pendulum angle

$$\frac{\partial L}{\partial \theta_i} = -mgl\sin(\theta)$$

- Partial derivative of Lagrangian wrt. pendulum velocity

$$\frac{\partial L}{\partial \dot{\theta}_i} = I\dot{\theta}$$

- Time derivative of Partial derivative of Lagrangian wrt. pendulum velocity

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} = \frac{d}{dt} I\dot{\theta} = I\ddot{\theta}$$

inertia remains constant

- Equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \tau_i$$

$$L = \frac{1}{2} I\dot{\theta}^2 - mgl(1-\cos\theta)$$

- Partial derivative of Lagrangian wrt. pendulum angle

$$\frac{\partial L}{\partial \theta_i} = -mgl\sin(\theta)$$

- Partial derivative of Lagrangian wrt. pendulum velocity

$$\frac{\partial L}{\partial \dot{\theta}_i} = I\dot{\theta}$$

- Time derivative of Partial derivative of Lagrangian wrt. pendulum velocity

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} = \frac{d}{dt} I\dot{\theta} = I\ddot{\theta}$$

- Equation of motion $I\ddot{\theta} + mgl\sin(\theta) = \tau_i$

$$L = \frac{1}{2}I\dot{\theta}^2 - mgl(1-\cos\theta)$$

- Partial derivative of Lagrangian wrt. pendulum angle $\frac{\partial L}{\partial \theta_i} =$

- Partial derivative of Lagrangian wrt. pendulum velocity $\frac{\partial L}{\partial \dot{\theta}_i} = I\dot{\theta}$

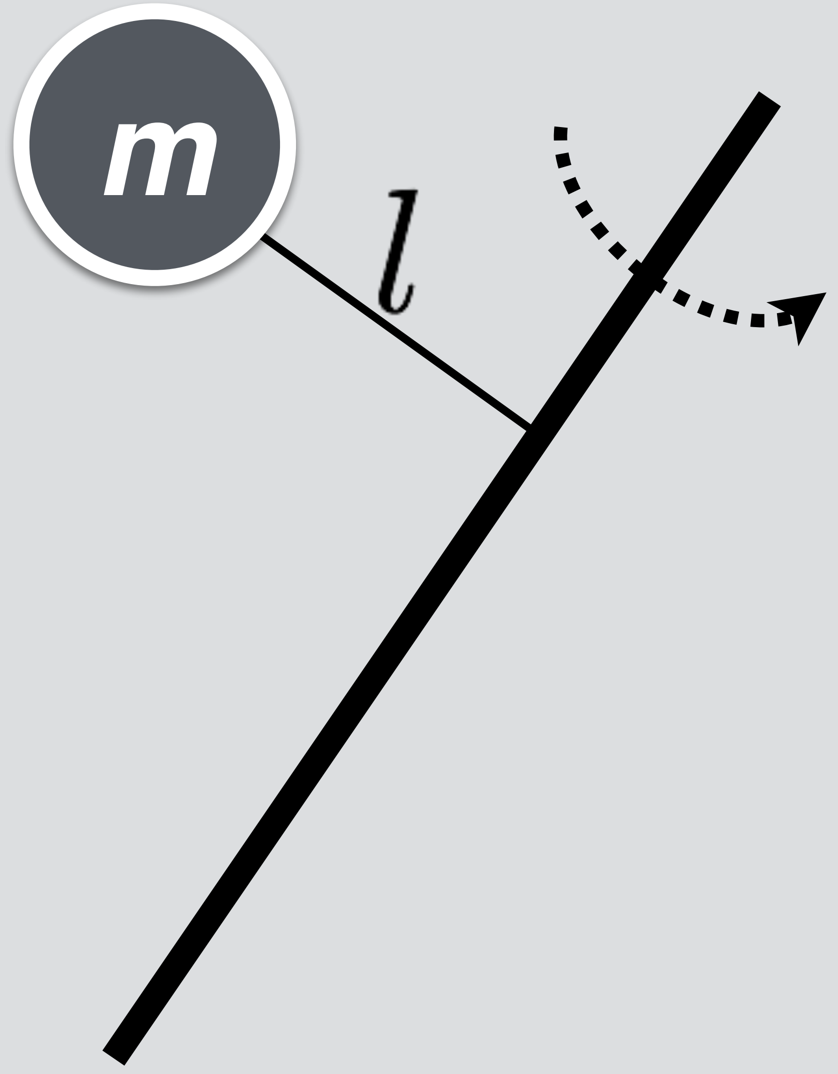
- Time derivative of Partial derivative of Lagrangian wrt. pendulum velocity $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} = \frac{d}{dt} I\dot{\theta} =$

- Equation of motion $I\ddot{\theta} + mgl\sin(\theta) = \tau_i$

$$L = \frac{1}{2}I\dot{\theta}^2 - mgl(1-\cos\theta)$$

- Parallel axis theorem $I = ml^2$

Inertia grows quadratically as mass moves further from its axis of rotation



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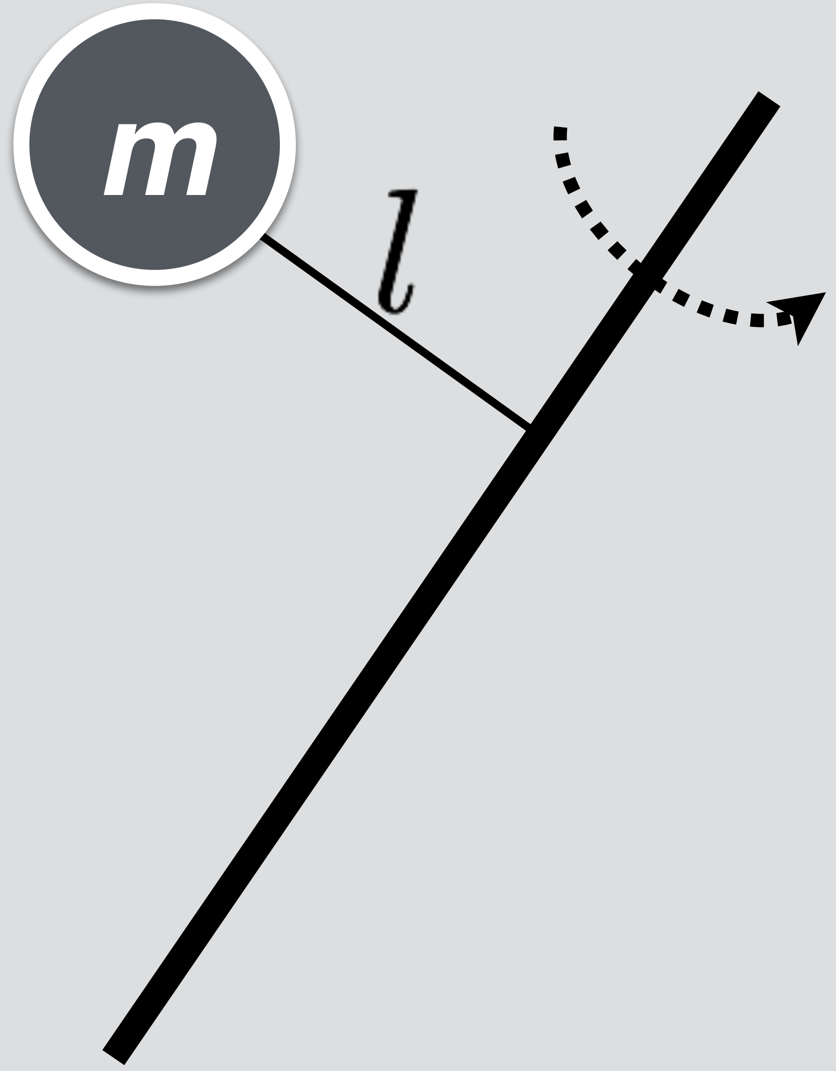
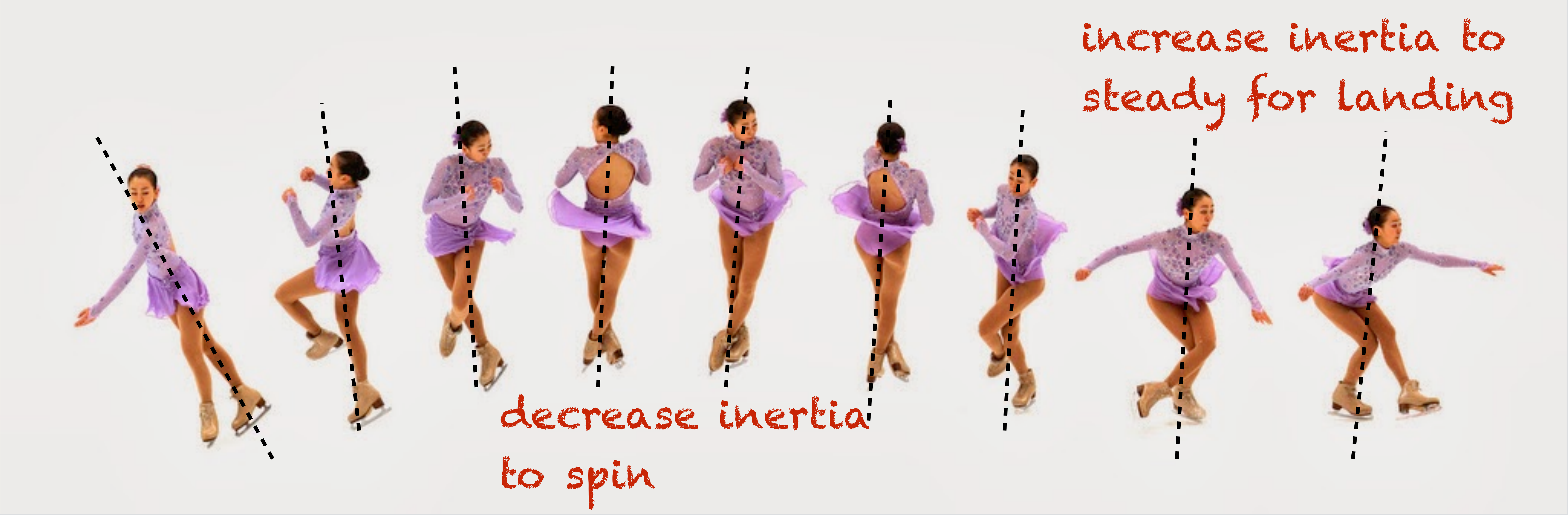


Inertia grows quadratically as mass moves further from its axis of rotation



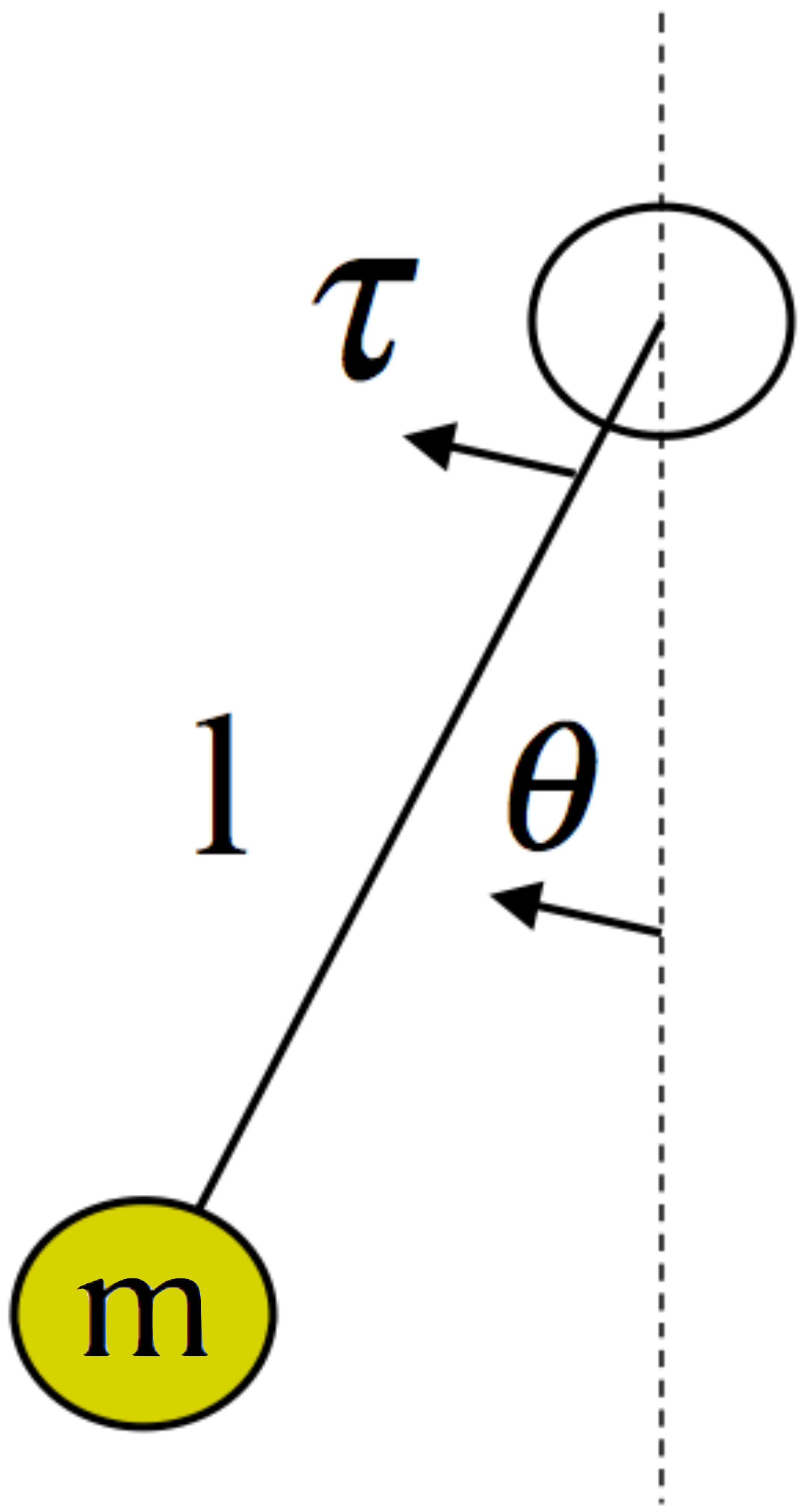
Example: Pendulum

Equation of motion
(with rotational inertia I)

$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$

with Parallel Axis Theorem ($I=ml^2$)

$$I\ddot{\theta} + mgl\sin(\theta) = \tau_i$$



Motor produces torque
(angular force)

Angle expresses pendulum
range of motion

Pendulum of length *l* with
point mass *m*

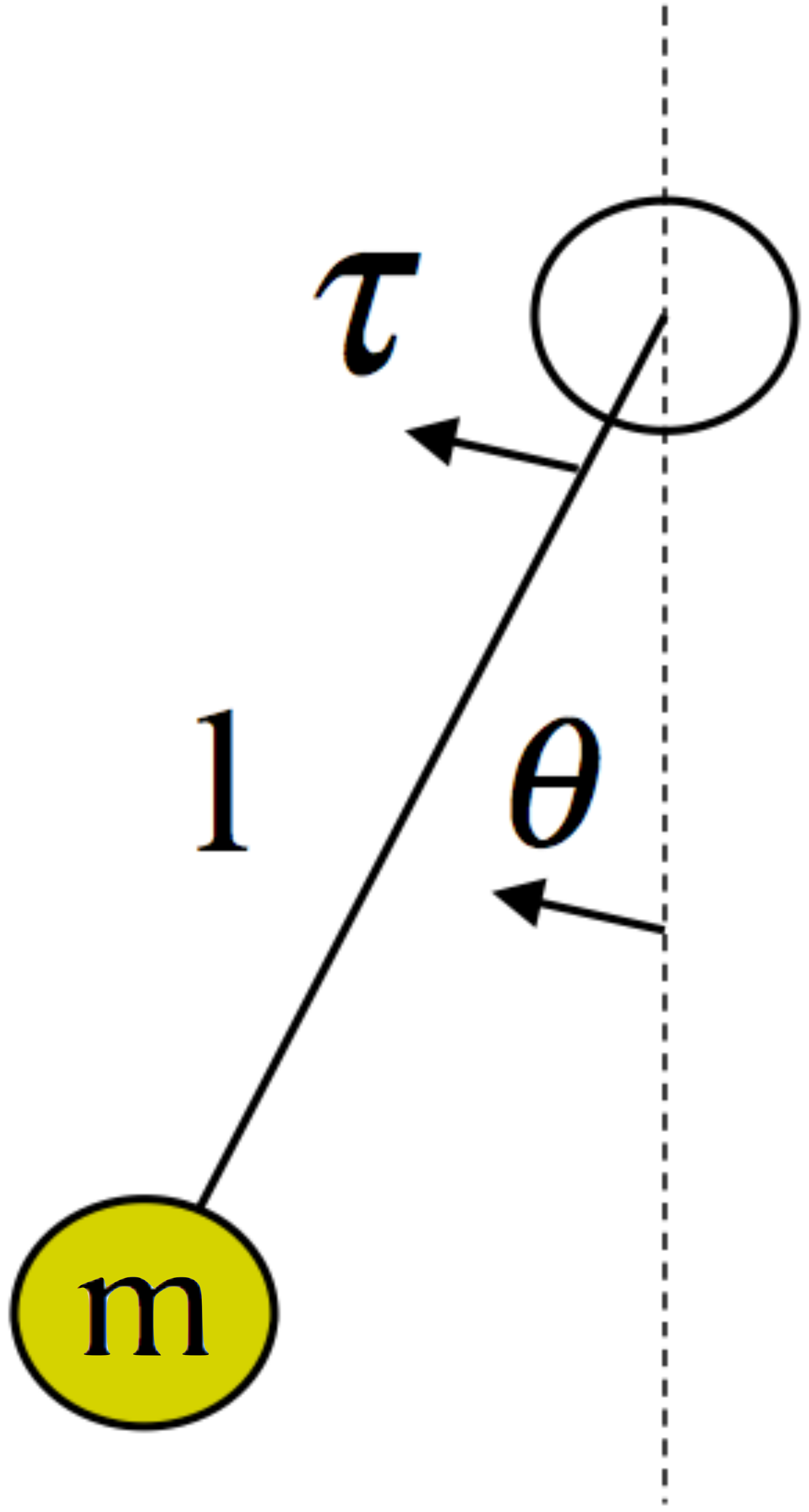
Example: Pendulum

Equation of motion
(with rotational inertia *I*)

$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$

with Parallel Axis Theorem ($I=ml^2$)

$$I\ddot{\theta} + mgl\sin(\theta) = \tau_i$$



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(with rotational inertia I)

$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$

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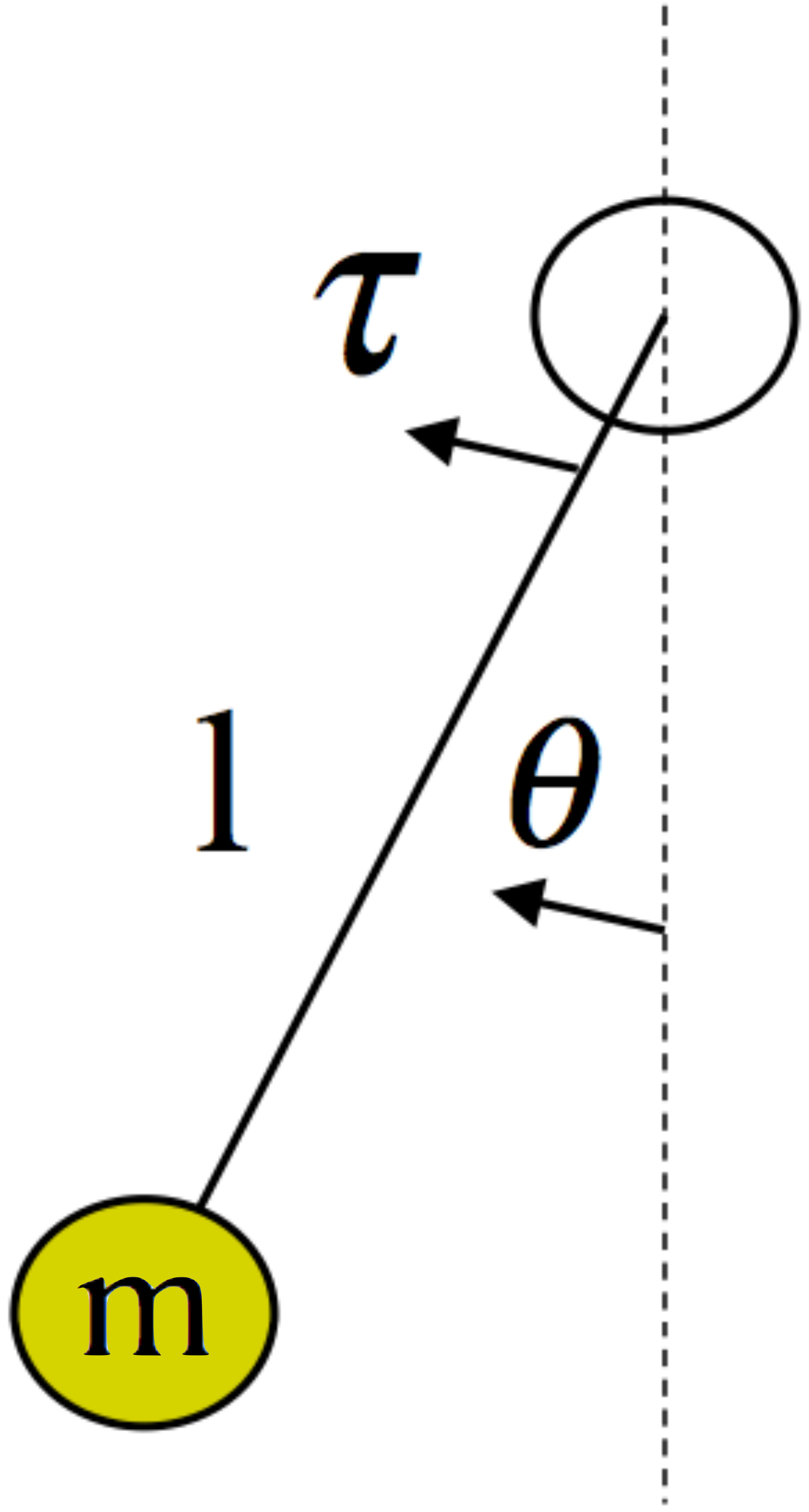
$$\ddot{\theta} = -\frac{g}{l}\sin(\theta) + \frac{\tau}{ml^2}$$



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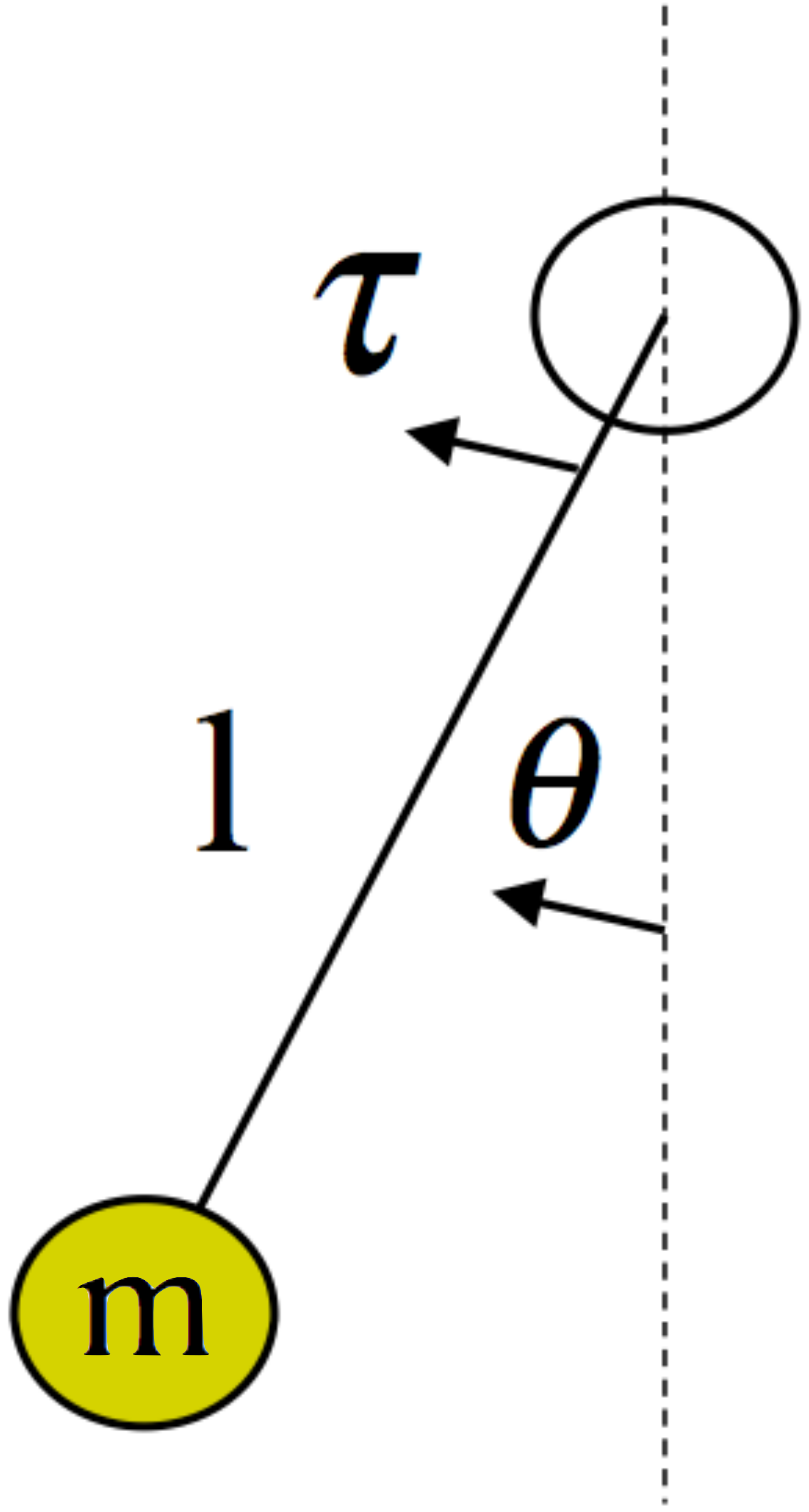
$$\ddot{\theta} = -\frac{g}{l}\sin(\theta) + \frac{\tau}{ml^2}$$

gravity term

motor term



Motor produces torque (angular force)

Angle expresses pendulum range of motion

Pendulum of length l with point mass m

Example: Pendulum

Equation of motion
(with rotational inertia I)

$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$

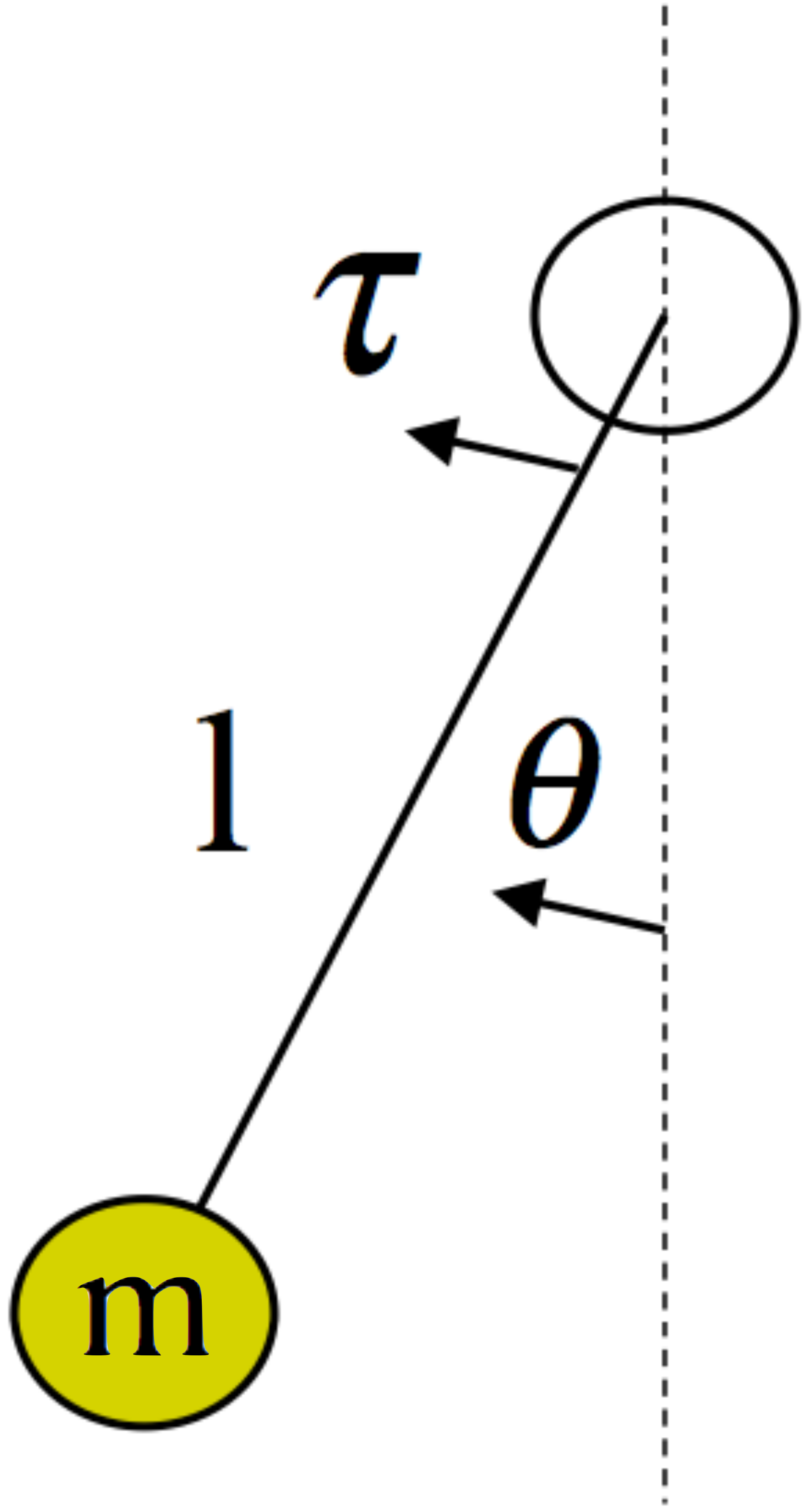
with Parallel Axis Theorem ($I=ml^2$)

$$\ddot{\theta} = -\frac{g}{l}\sin(\theta) + \frac{\tau}{ml^2}$$

Numerical integration over time

$$\theta_{t+\Delta t} = \theta_t + \dot{\theta}_t \Delta t$$

$$\dot{\theta}_{t+\Delta t} = \dot{\theta}_t + \ddot{\theta}_t \Delta t$$



Motor produces torque
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Angle expresses pendulum
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Pendulum of length *l* with
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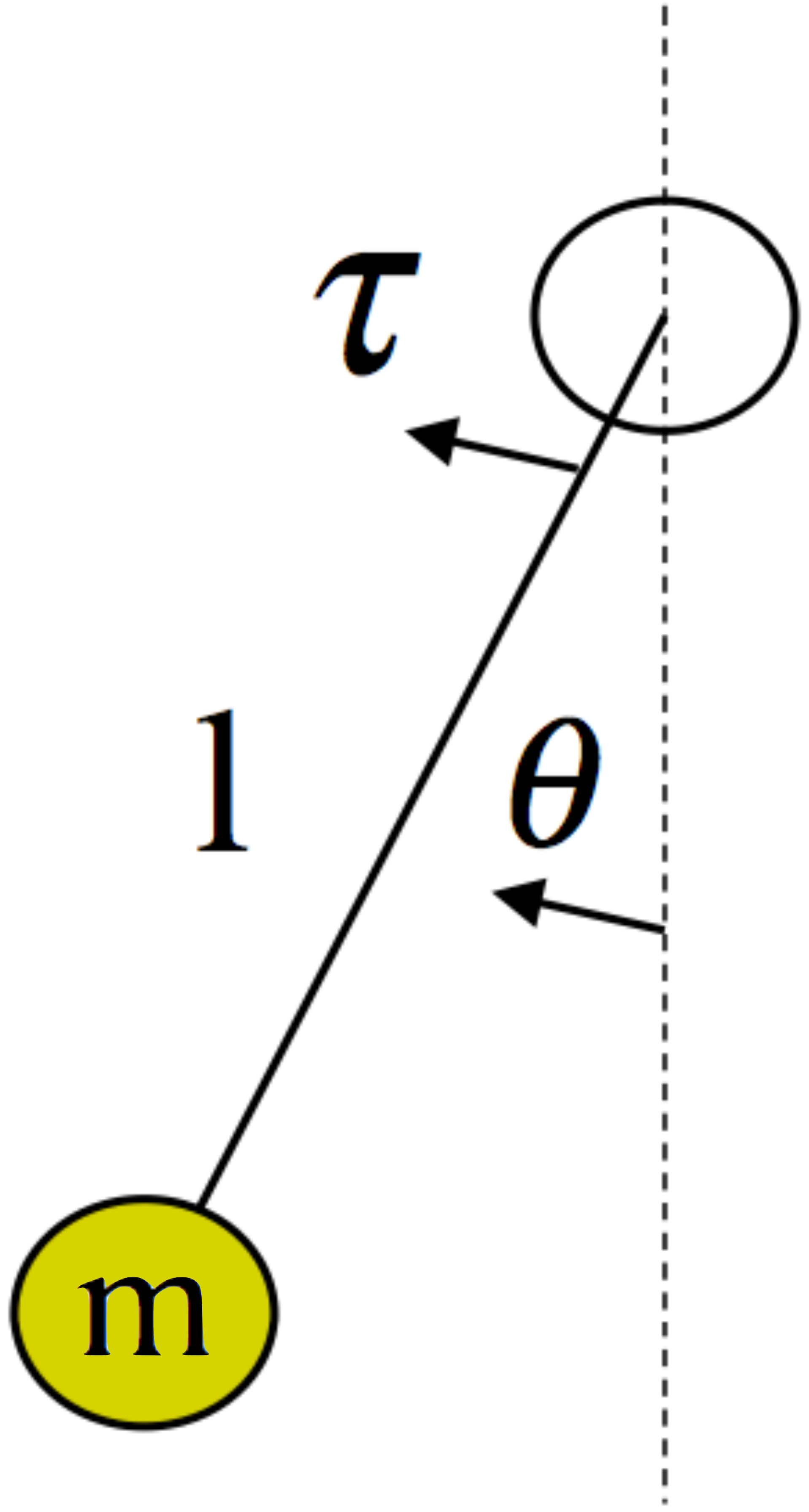


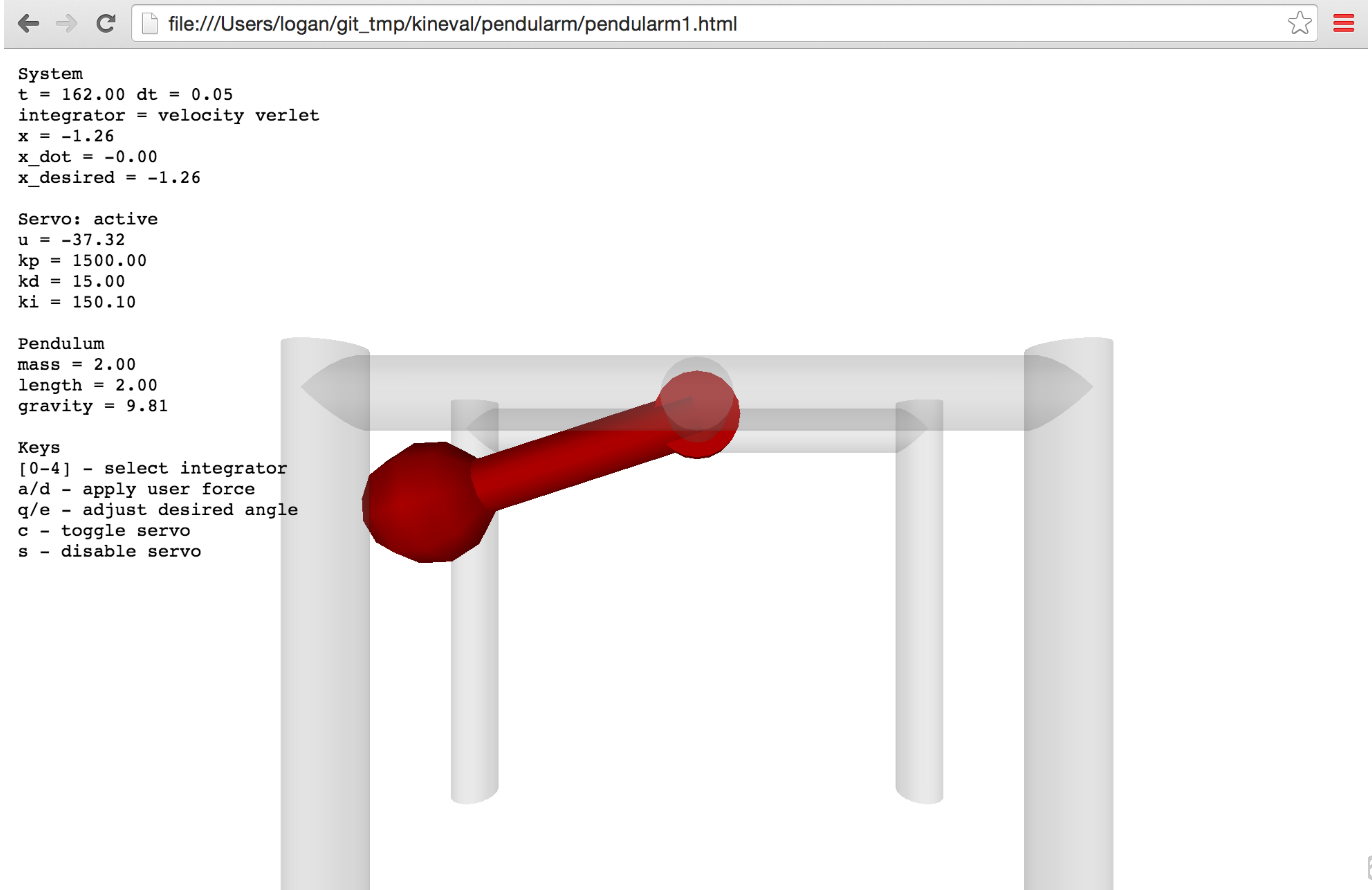
Euler integration

Motor produces torque
(angular force)

Note: state is both pendulum angle and velocity

Pendulum of length *l* with
point mass *m*

Let’s see what happens



Why is Euler Integration not the best choice?

Why is Euler Integration not the best choice?

Think about it as an initial value problem

Initial value problem

From Wikipedia, the free encyclopedia

In mathematics, in the field of differential equations, an **initial value problem** (also called **the Cauchy problem** by some authors) is an ordinary differential equation together with a specified value, called the **initial condition**, of the unknown function at a given point in the domain of the solution. In physics or other sciences, modeling a system frequently amounts to solving an initial value problem; in this context, the differential equation is an evolution equation specifying how, given initial conditions, the system will evolve with time.

Initial value problem

From Wikipedia, the free encyclopedia

Given initial condition (t_0, y_0) where $y_0 = f(t_0)$

time

current
state

Initial value problem

From Wikipedia, the free encyclopedia

Given initial condition (t_0, y_0) where $y_0 = f(t_0)$ and differential equation $y' = \frac{dy}{dt}$ in the form

$$y'(t) = f(t, y(t))$$

system velocity

system dynamics

Initial value problem

From Wikipedia, the free encyclopedia

Given initial condition (t_0, y_0) where $y_0 = f(t_0)$ and

differential equation $y' = \frac{dy}{dt}$ in the form $y'(t) = f(t, y(t))$

How to estimate future state $y(t_0 + h)$?

next
state

Initial value problem

From Wikipedia, the free encyclopedia

Given initial condition (t_0, y_0) where $y_0 = f(t_0)$ and

differential equation $y' = \frac{dy}{dt}$ in the form $y'(t) = f(t, y(t))$

How to estimate future state $y(t_0 + h)$?

numerical integration over timestep duration

$$y(t_0 + h) - y(t_0) = \int_{t_0}^{t_0+h} f(t, y(t))dt$$

next state

current state

integration over timestep

Euler Integration

Integral over timestep approximated as $\int_{t_0}^{t_0+h} f(t, y(t))dt \approx hf(t_0, y(t_0))$

Discrete step from iteration *n* to iteration *n+1* computed in two assignments

Advance state

$$y_{n+1} = y_n + hf(t_n, y_n)$$

next state

current state

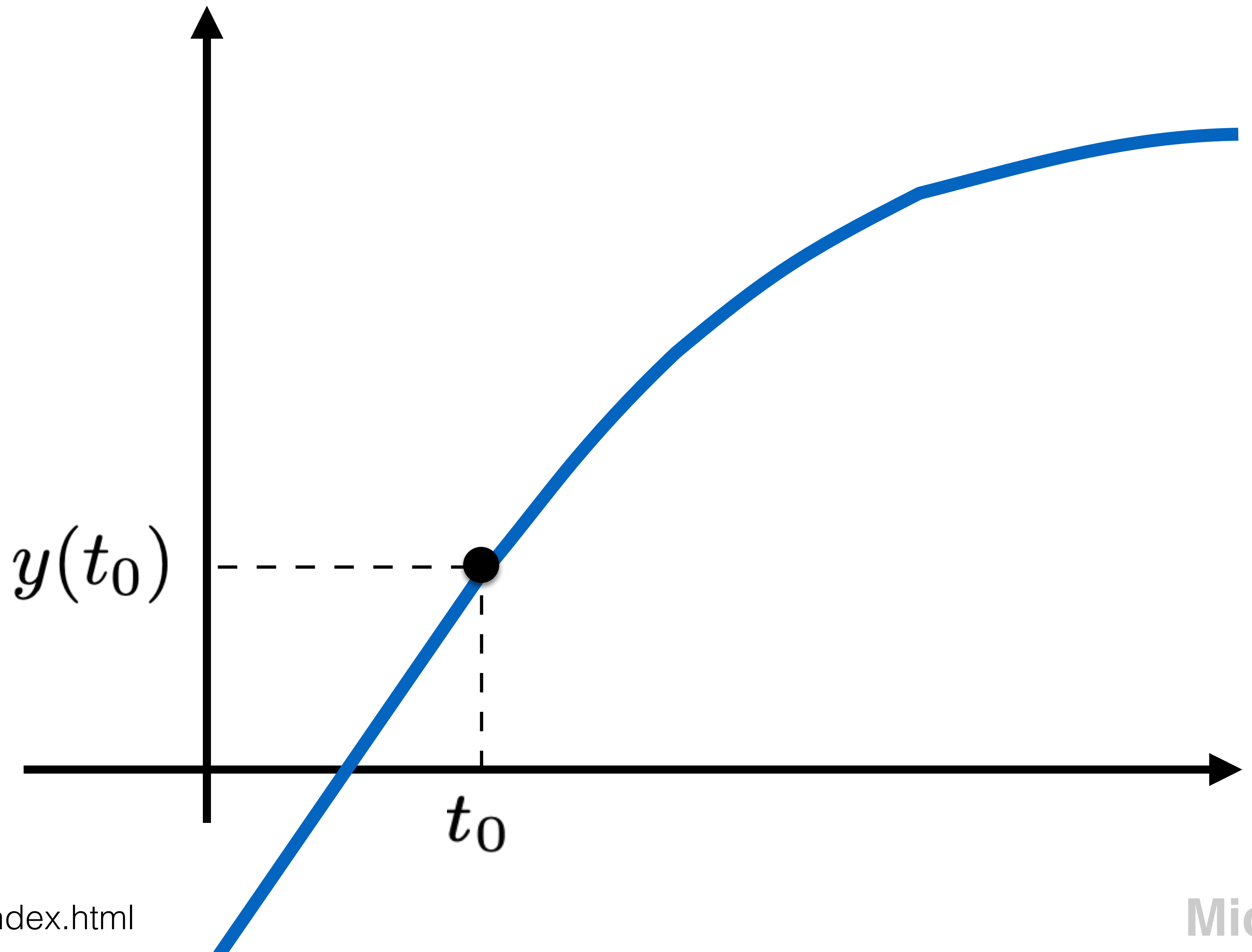
timestep duration * initial rate of change

Advance time

$$t_{n+1} = t_n + h$$

Euler Integration

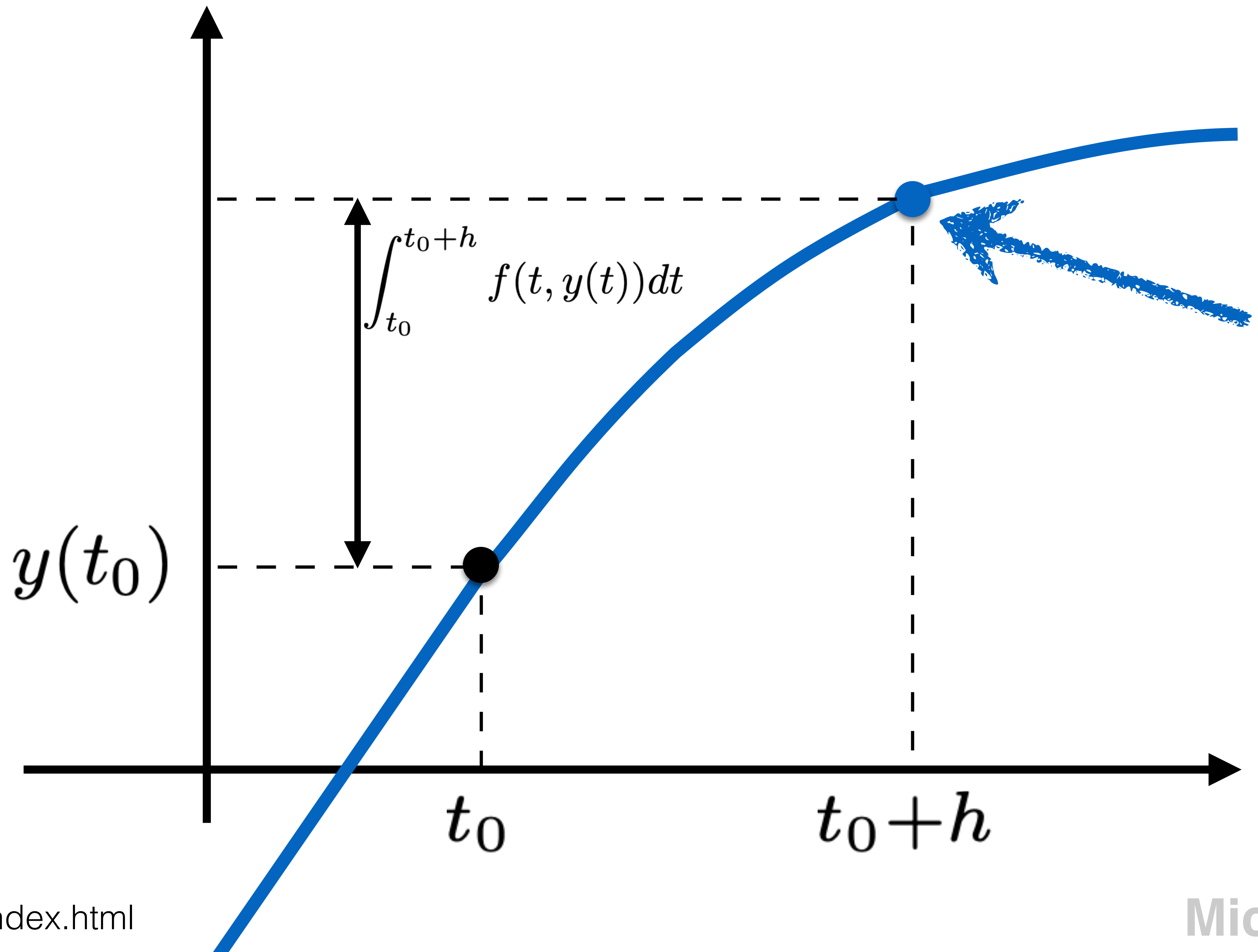
Integral over timestep approximated as $\int_{t_0}^{t_0+h} f(t, y(t))dt \approx hf(t_0, y(t_0))$



http://astro.physics.ncsu.edu/urca/course_files/Lesson15/index.html

Euler Integration

Integral over timestep approximated as $\displaystyle\int_{t_0}^{t_0+h} f(t, y(t))dt \approx hf(t_0, y(t_0))$

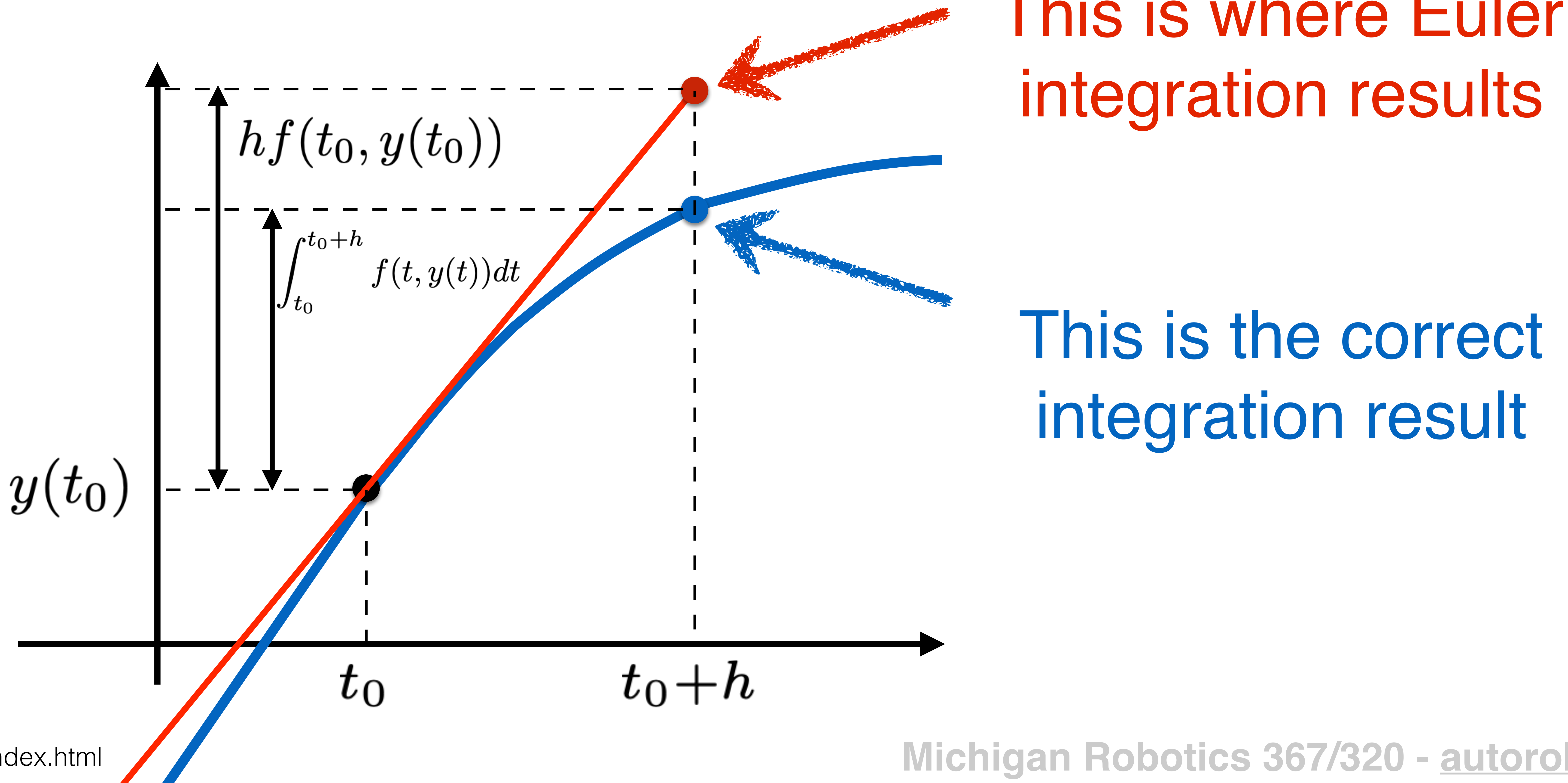


This is the correct integration result

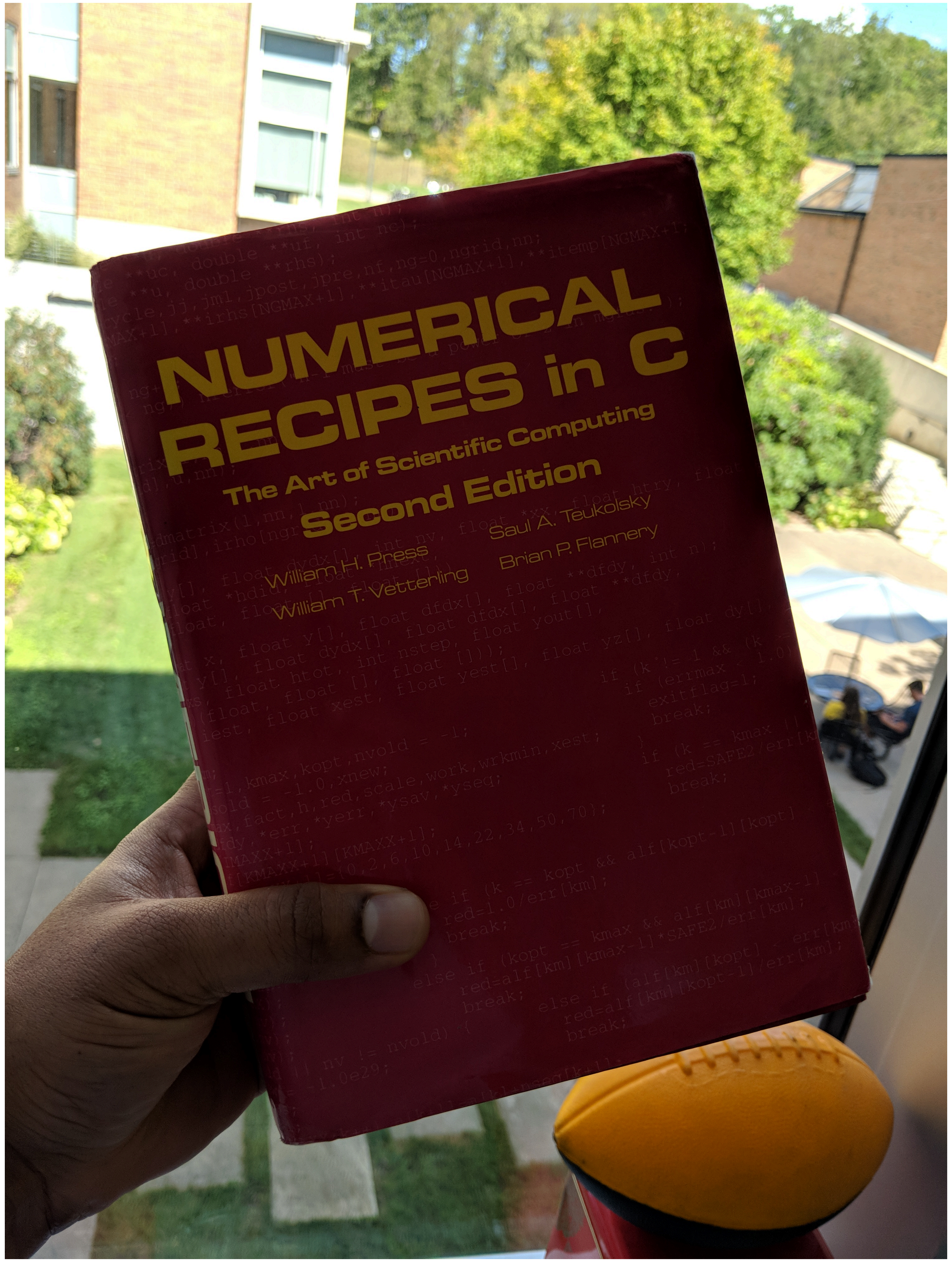
http://astro.physics.ncsu.edu/urca/course_files/Lesson15/index.html

Euler Integration

Integral over timestep approximated as $\int_{t_0}^{t_0+h} f(t, y(t))dt \approx hf(t_0, y(t_0))$



http://astro.physics.ncsu.edu/urca/course_files/Lesson15/index.html



The unofficial textbook of this lecture

Example Euler integration of 2D point over 2 time steps

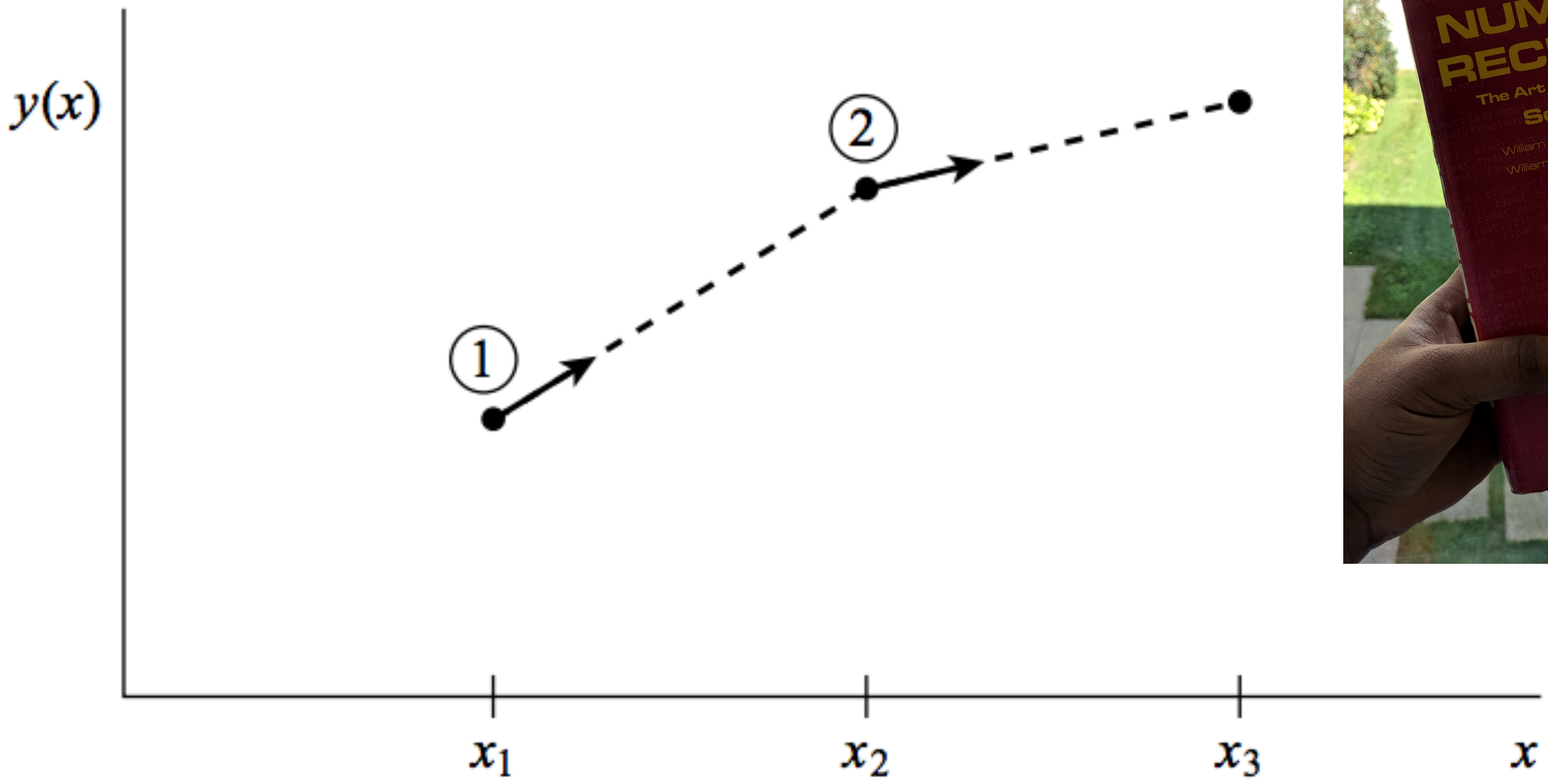
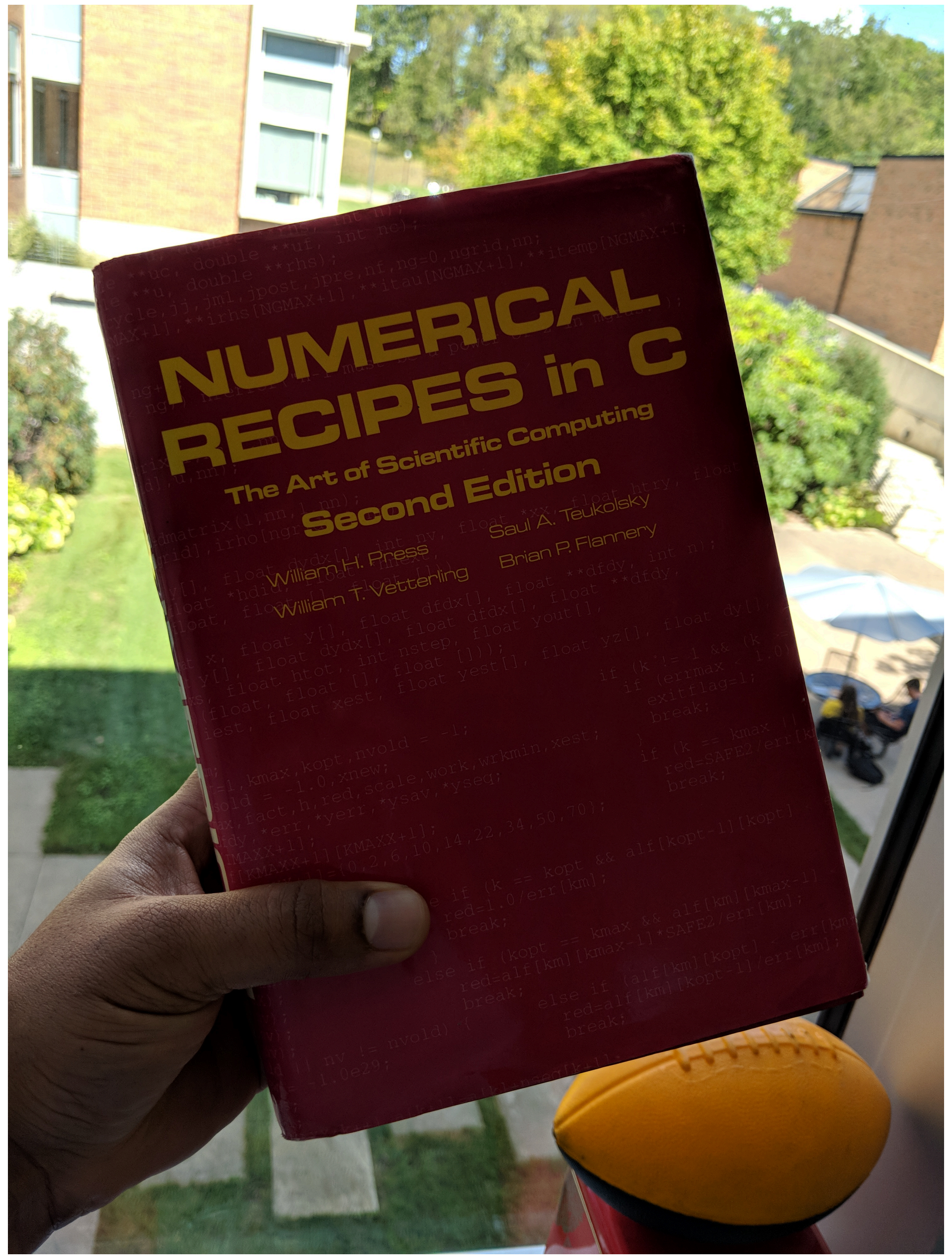


Figure 16.1.1. Euler's method. In this simplest (and least accurate) method for integrating an ODE, the derivative at the starting point of each interval is extrapolated to find the next function value. The method has first-order accuracy.



Second-order state

Euler's method

Advance position using velocity

$$y_{n+1} = y_n + \dot{y}_n \Delta t$$

Advance velocity using acceleration

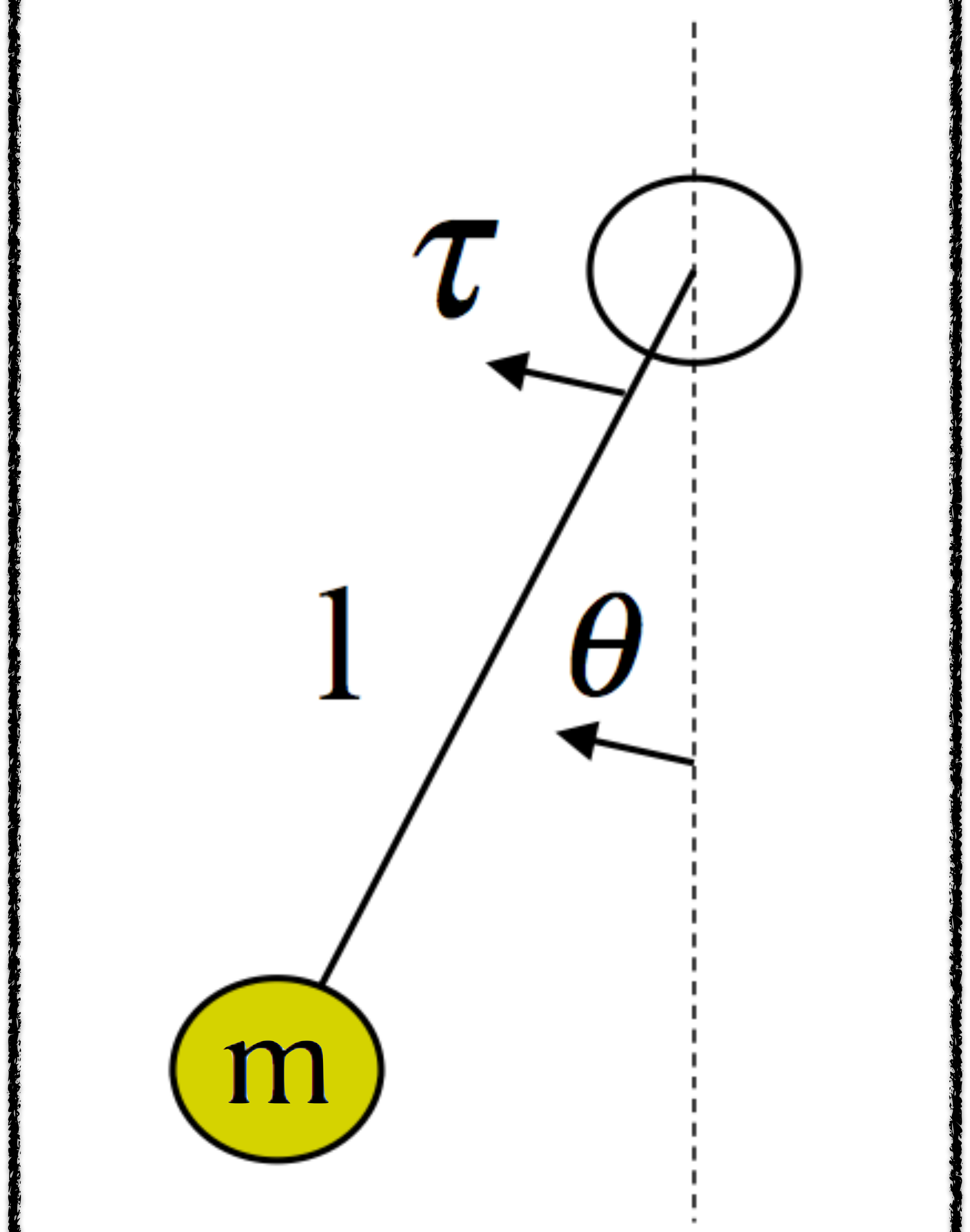
$$\dot{y}_{n+1} = \dot{y}_n + \ddot{y}_n \Delta t$$

$a(y_n)$

acceleration

Advance time

$$t_{n+1} = t_n + \Delta t$$



$$\ddot{\theta} = -\frac{g}{l}\sin(\theta) + \frac{\tau}{ml^2}$$

assumes second order system f(y,y′,y″) treats velocity y′ = f(y) and acceleration y″ = f′(y′) individually

Euler's method

Advance position using velocity

$$y_{n+1} = y_n + \dot{y}_n \Delta t$$

Advance velocity using acceleration

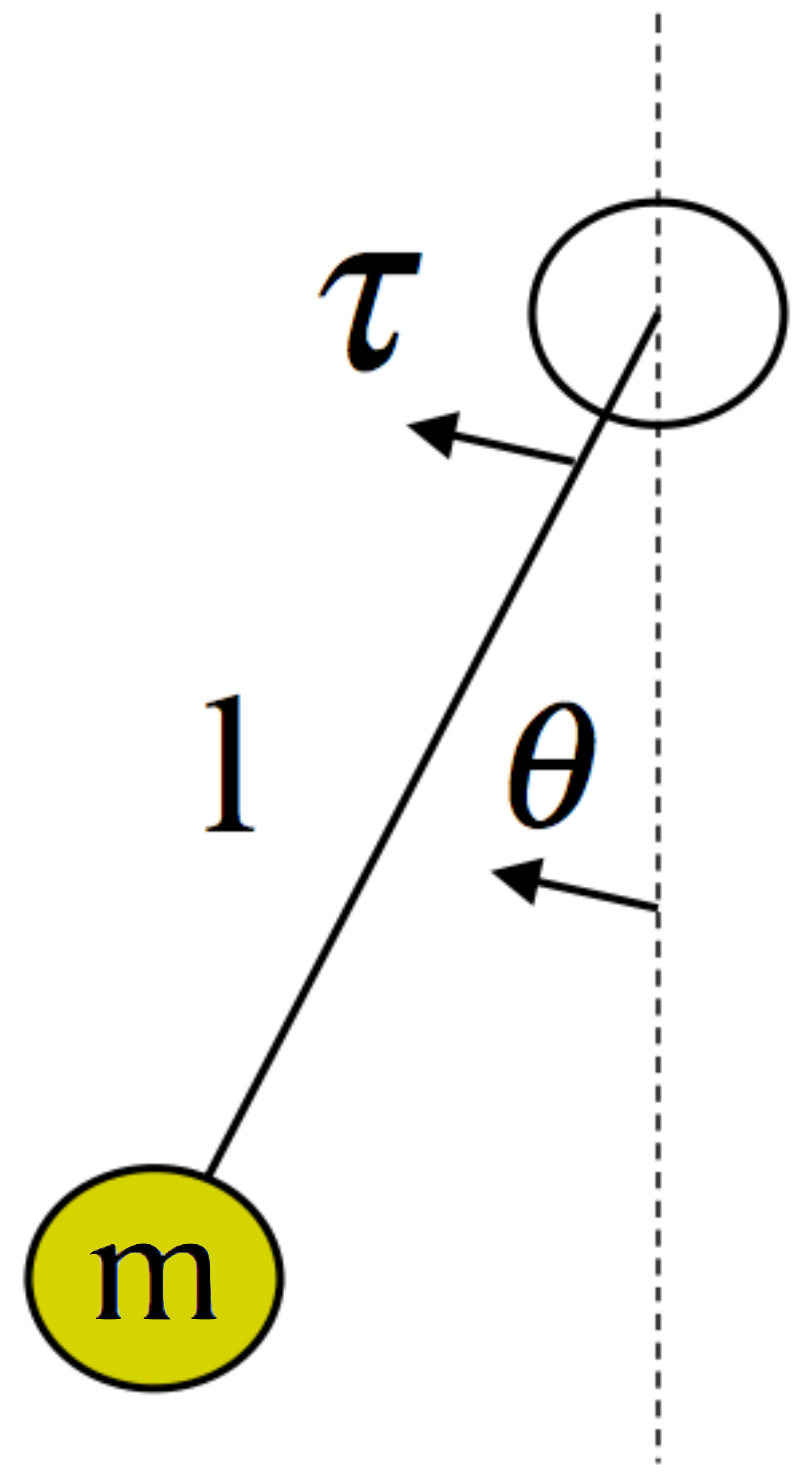
$$\dot{y}_{n+1} = \dot{y}_n + \ddot{y}_n \Delta t$$

$a(y_n)$

acceleration

Advance time

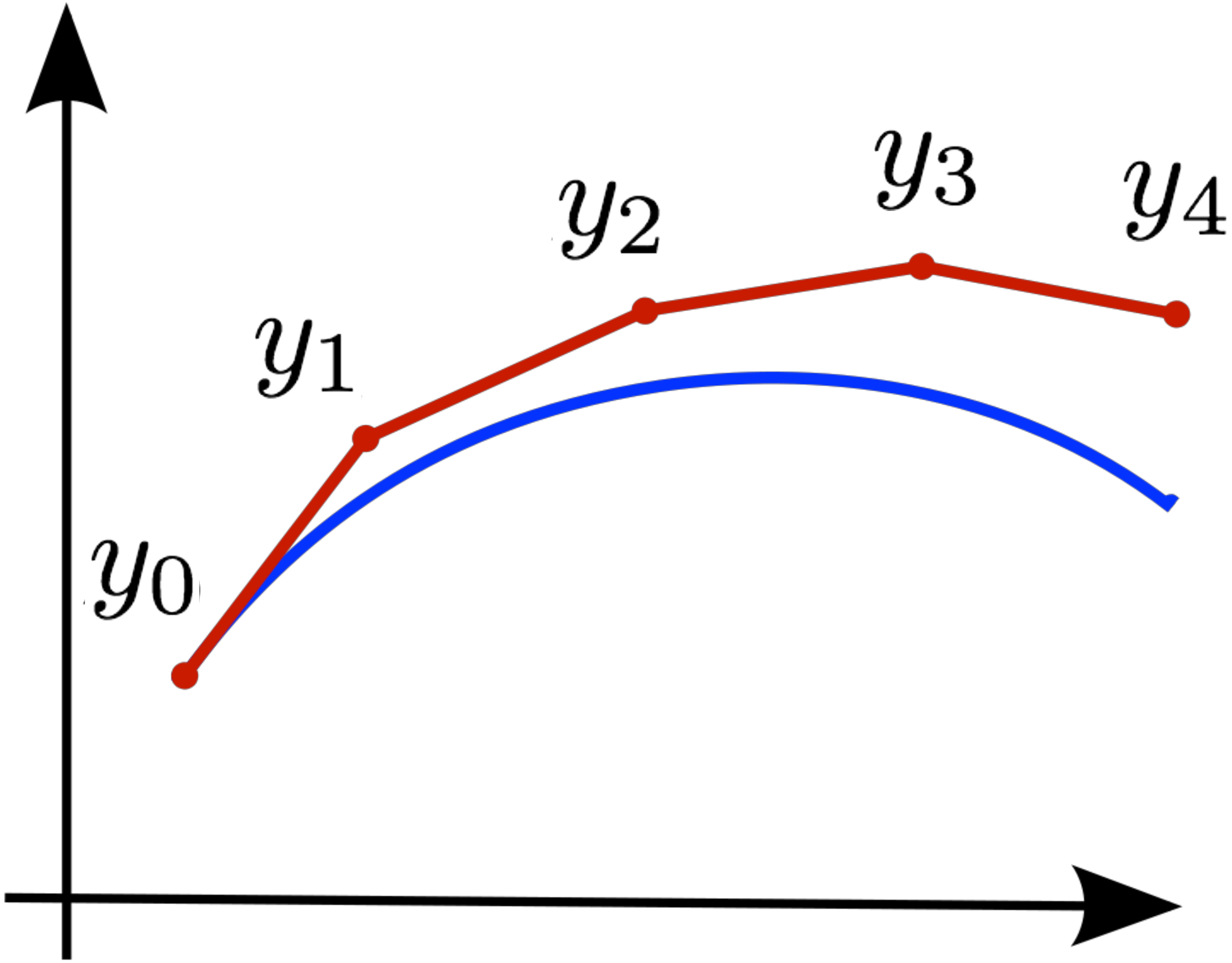
$$t_{n+1} = t_n + \Delta t$$



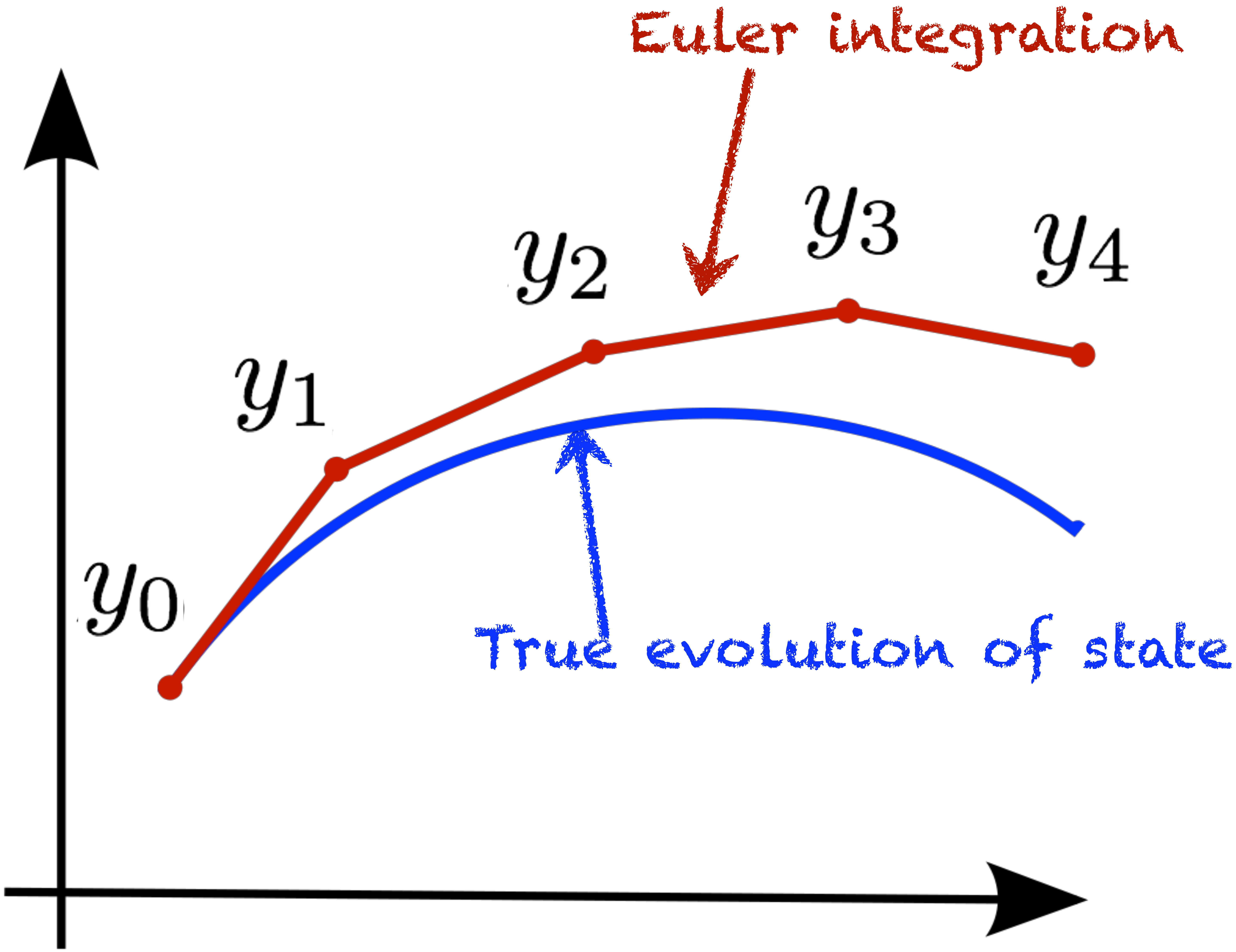
$$\ddot{\theta} = -\frac{g}{l}\sin(\theta) + \frac{\tau}{ml^2}$$

Why is the pendulum going unstable?

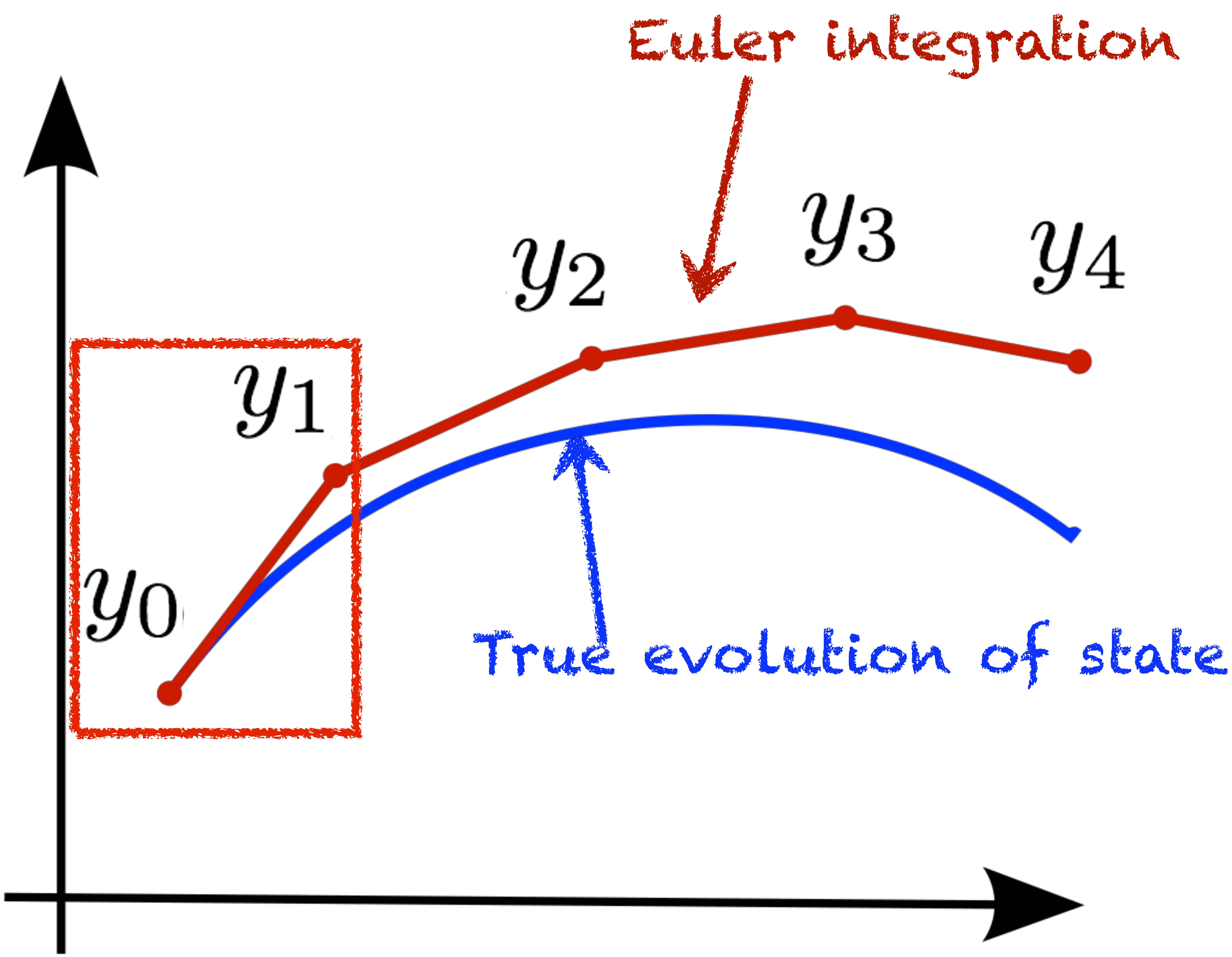
Example Euler integration of 2D point over 4 time steps

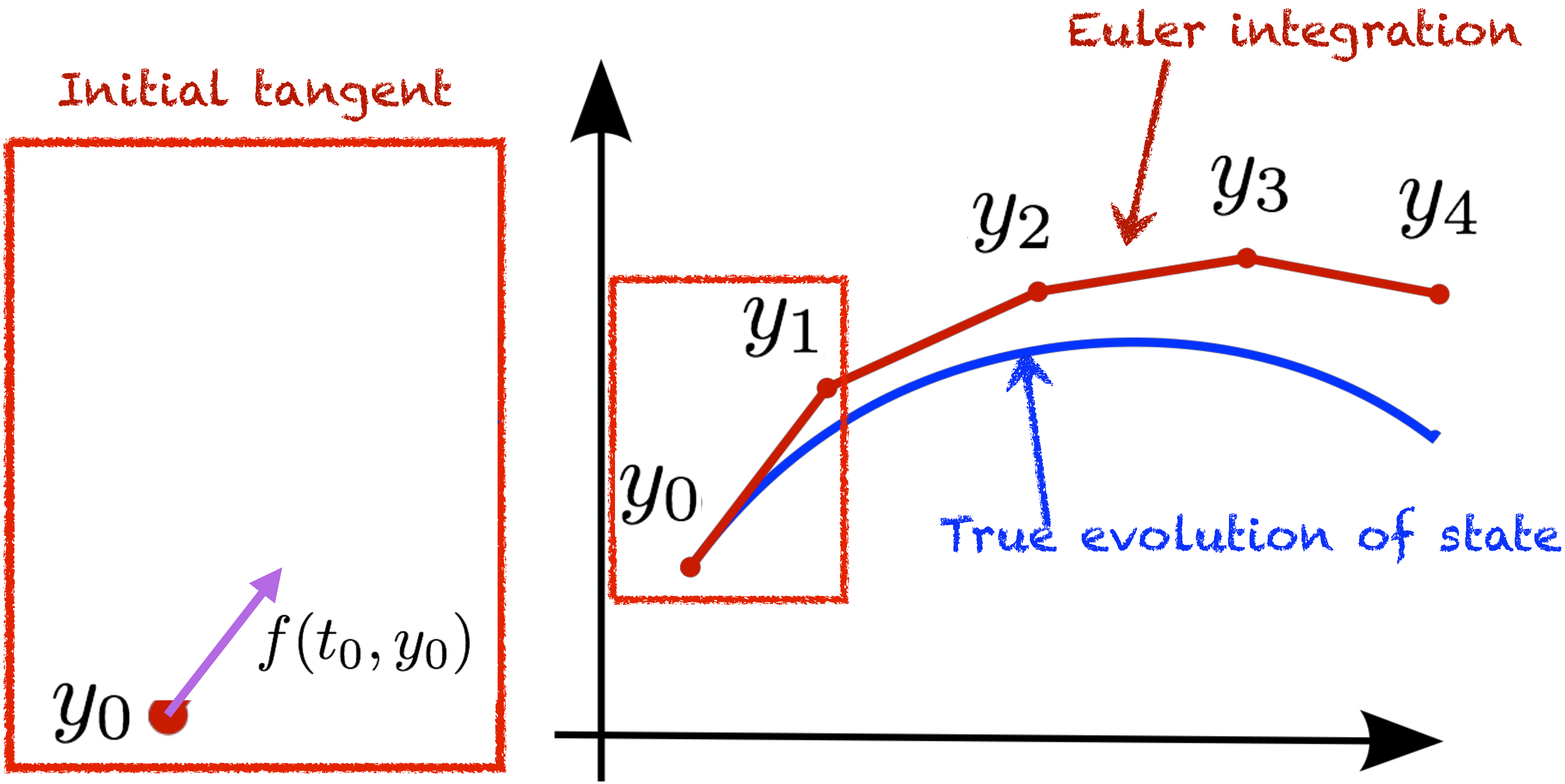


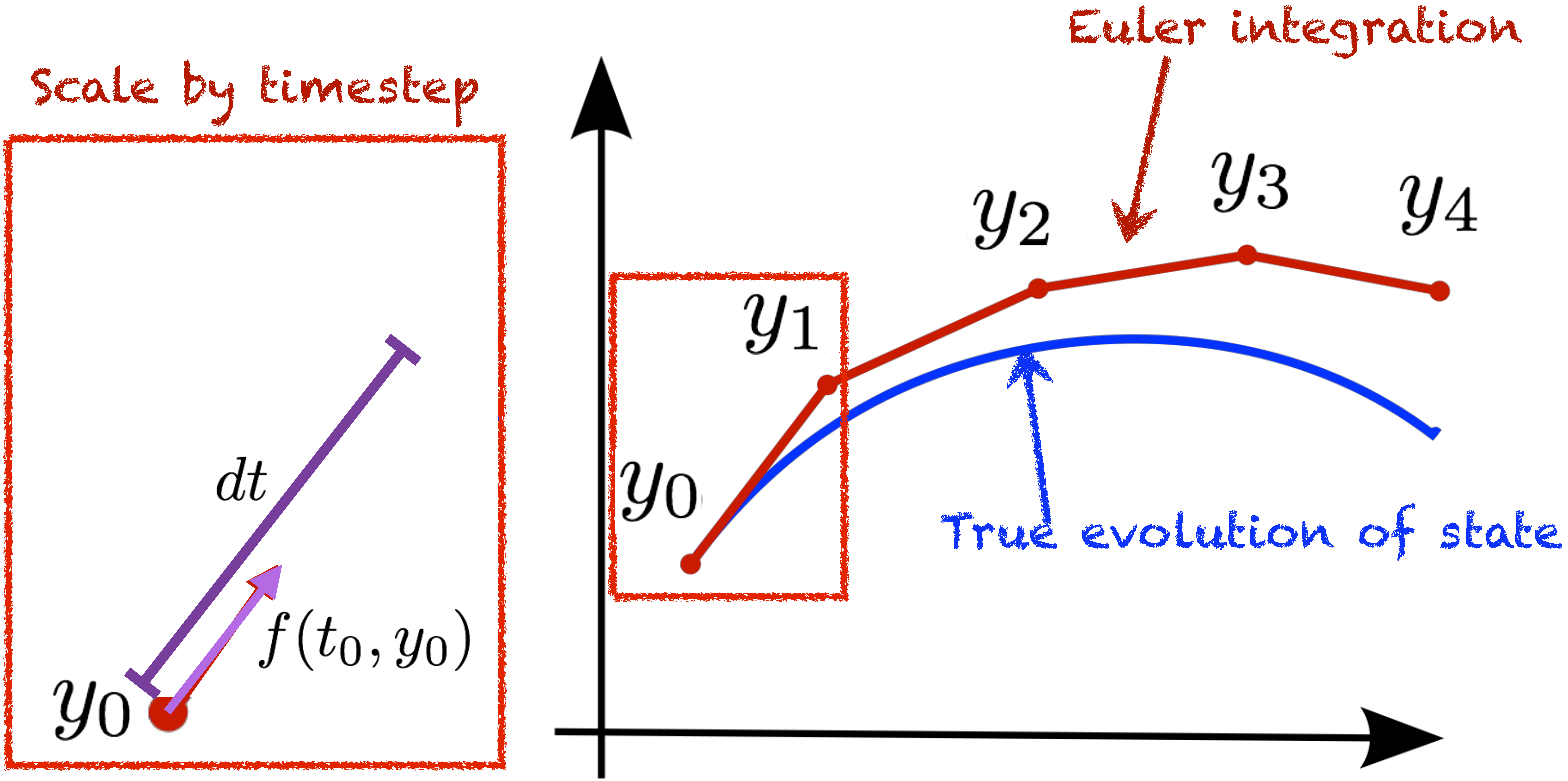
Example Euler integration of 2D point over 4 time steps



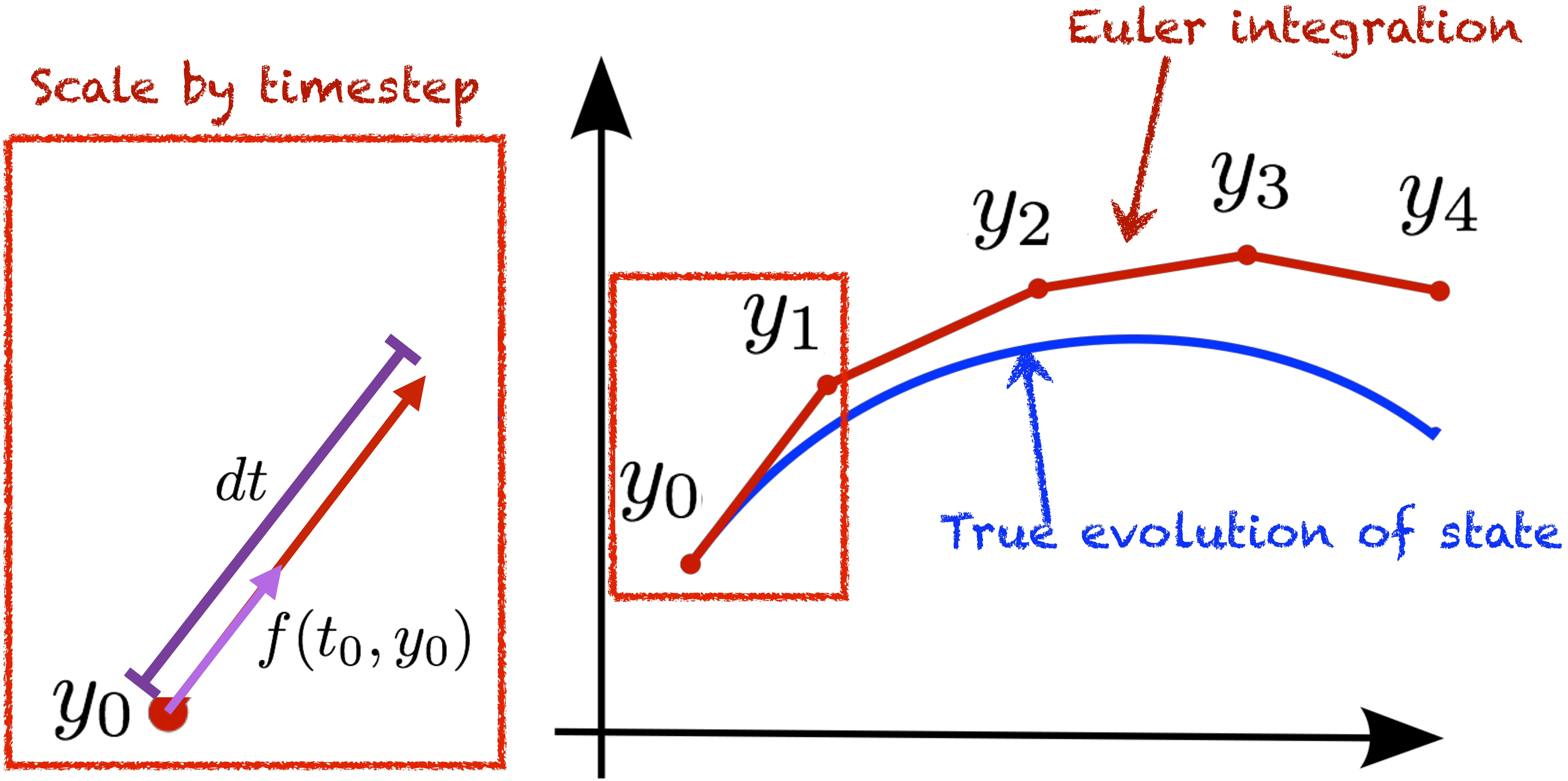
Example Euler integration of 2D point over 4 time steps

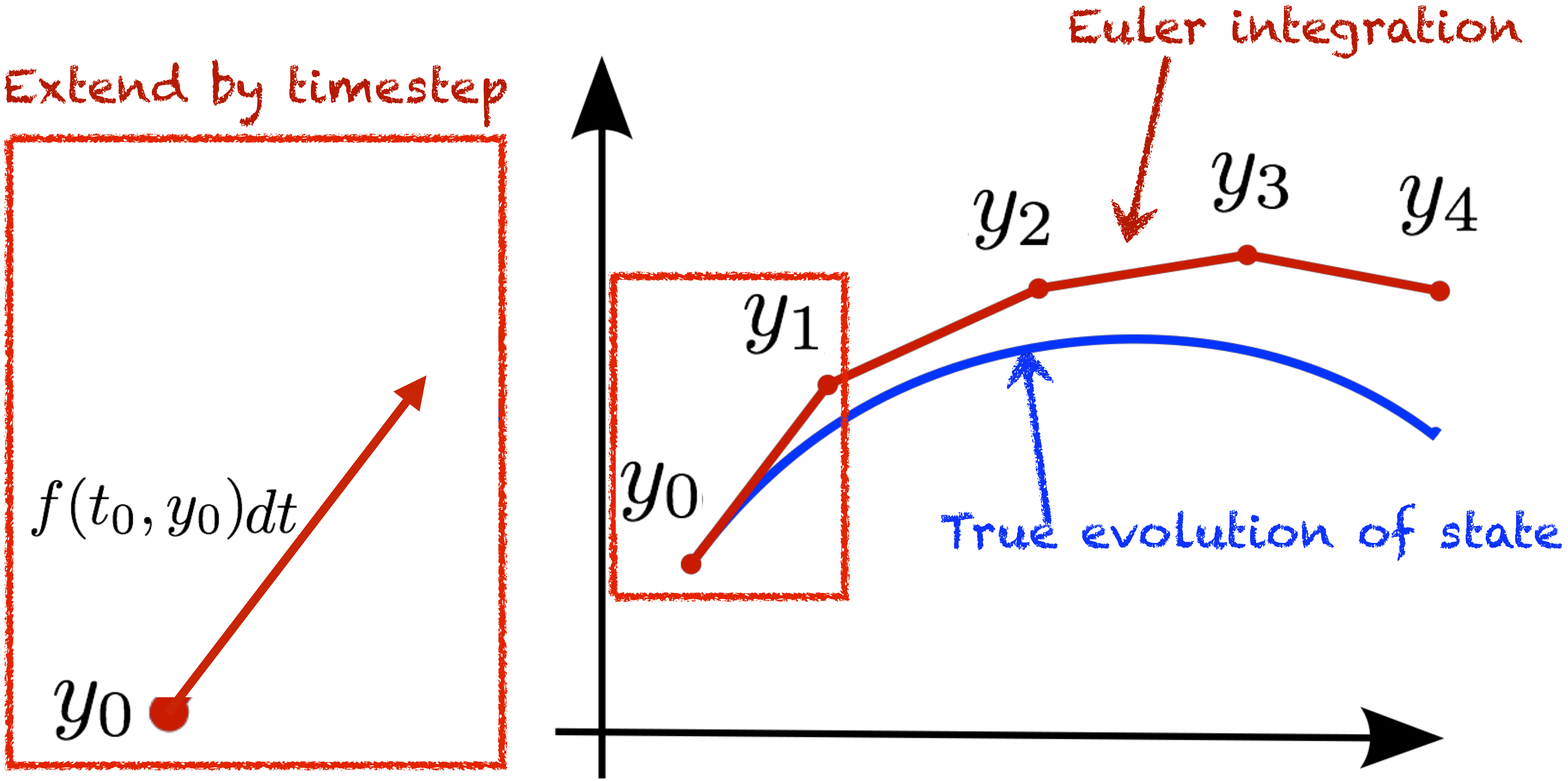


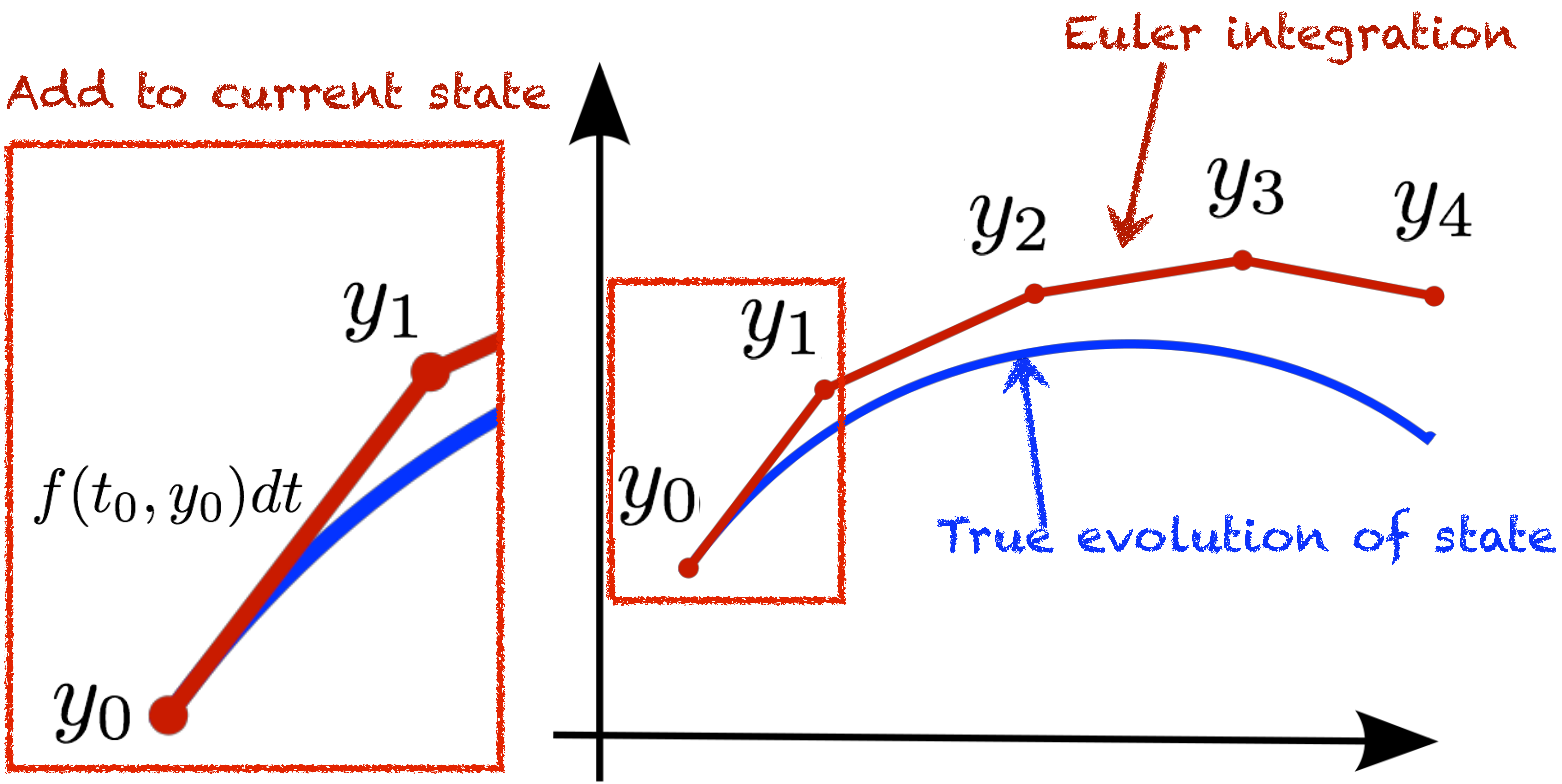

Initial tangent
y_0
$f(t_0, y_0)$
Euler integration
y_0
y_1
y_2
y_3
y_4
True evolution of state

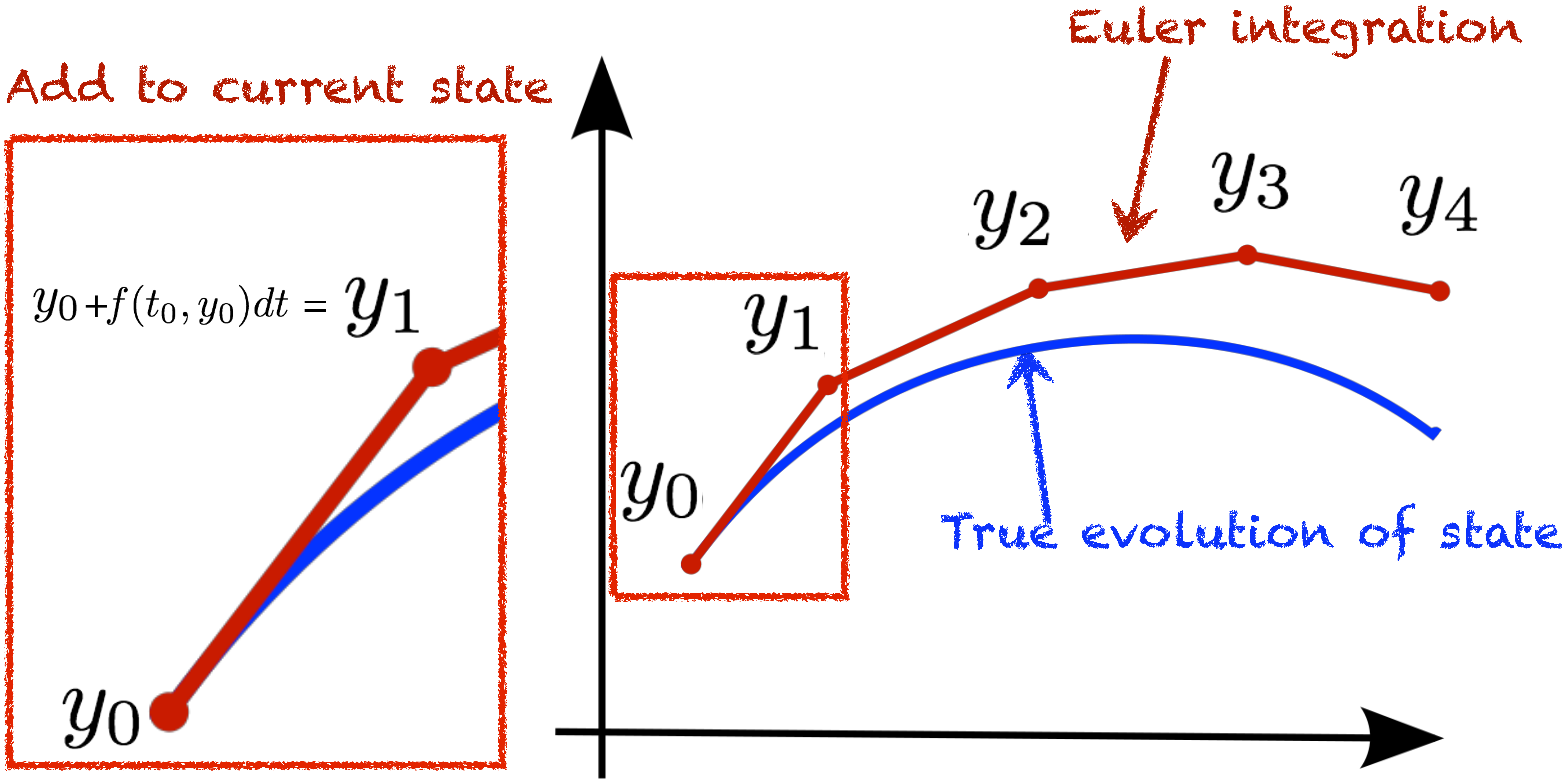


Scale by timestep
Euler integration
dt
$f(t_0, y_0)$
y_0
y_0
y_1
y_2
y_3
y_4
True evolution of state

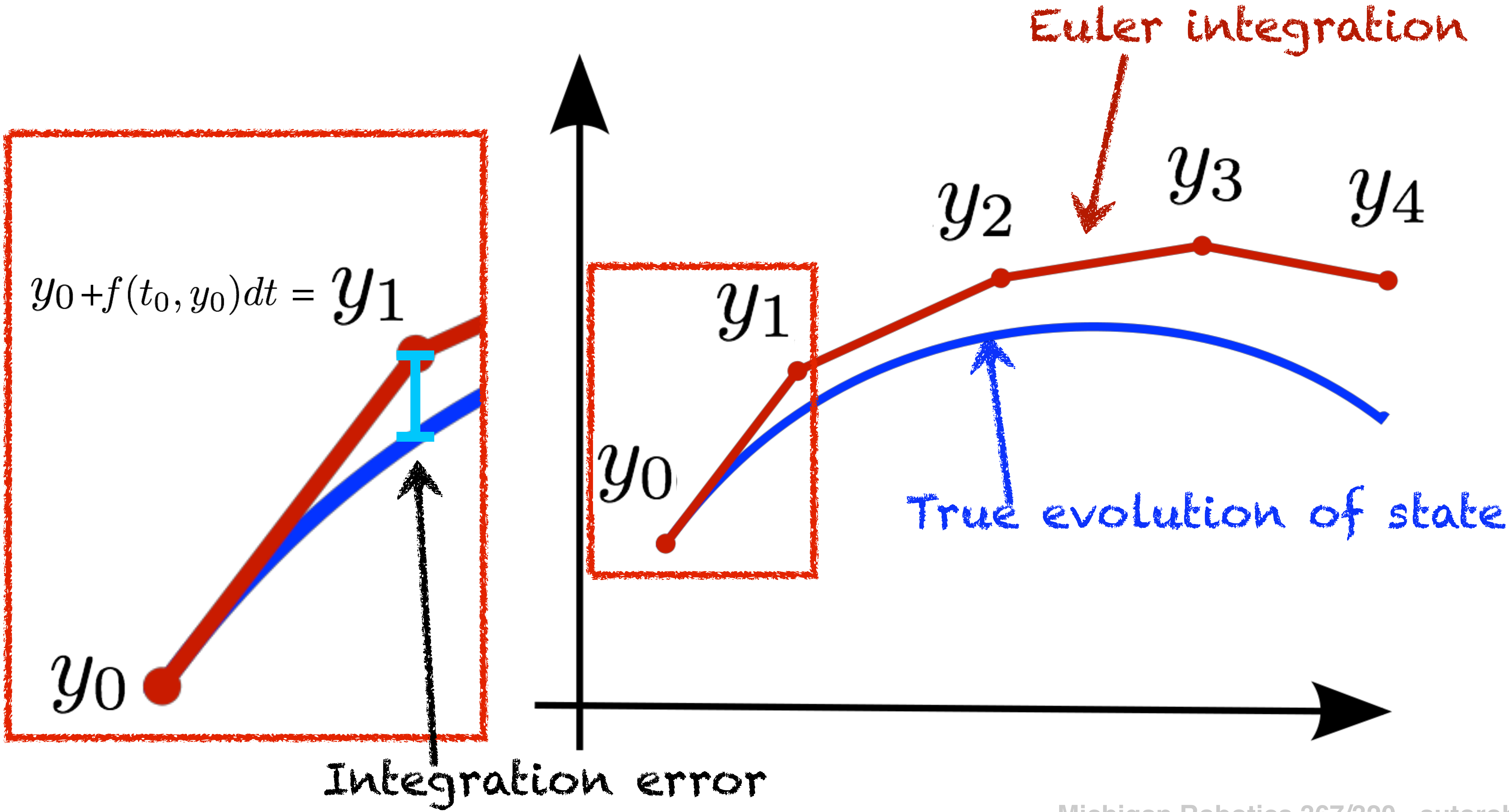

Scale by timestep
dt
$f(t_0, y_0)$
y_0
Euler integration
y_0
y_1
y_2
y_3
y_4
True evolution of state


Extend by timestep
$f(t_0, y_0)dt$
y_0
Euler integration
y_0
y_1
y_2
y_3
y_4
True evolution of state

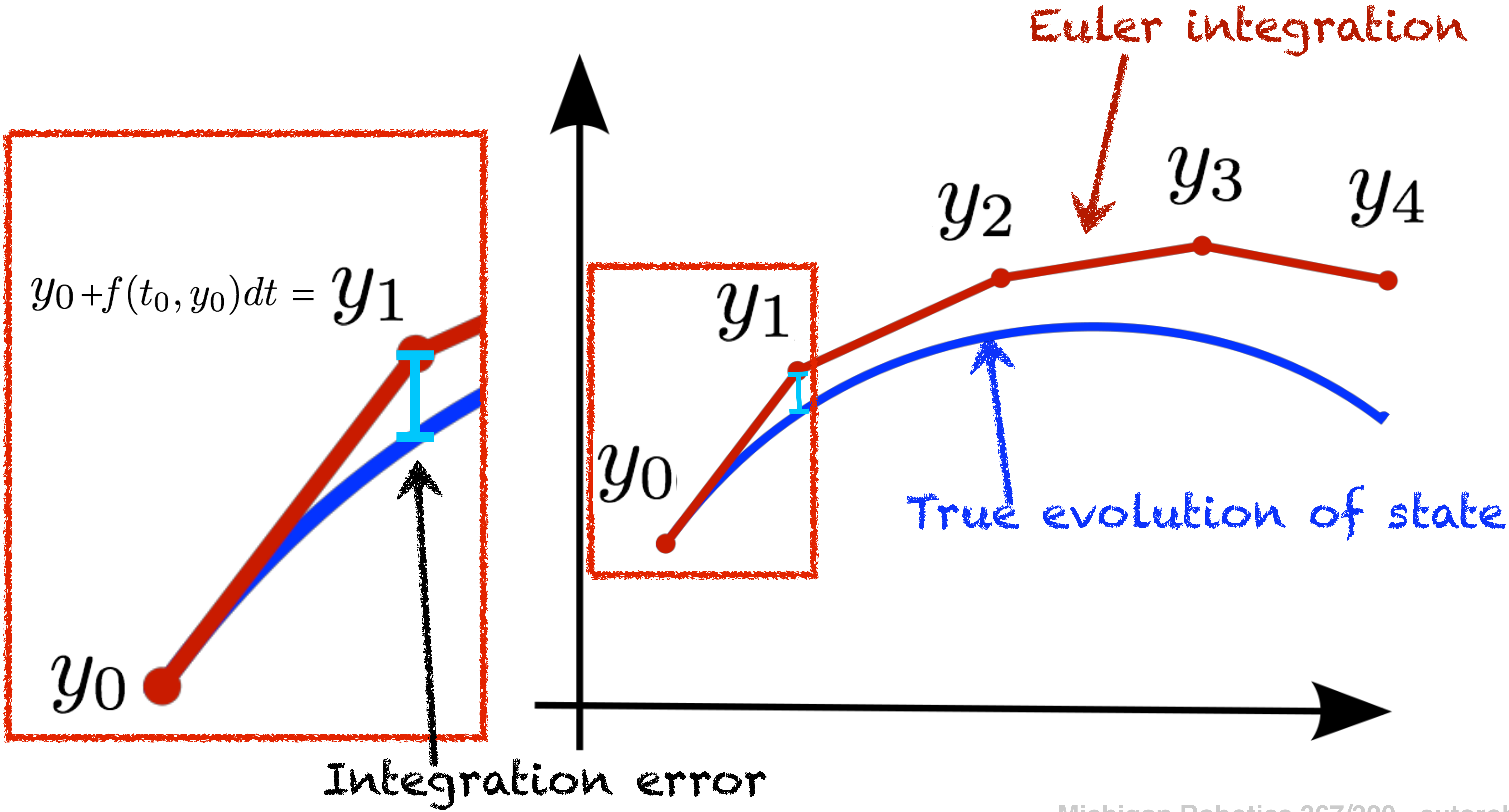

Add to current state
Euler integration
y_1
$f(t_0, y_0)dt$
y_0
y_2
y_3
y_4
True evolution of state



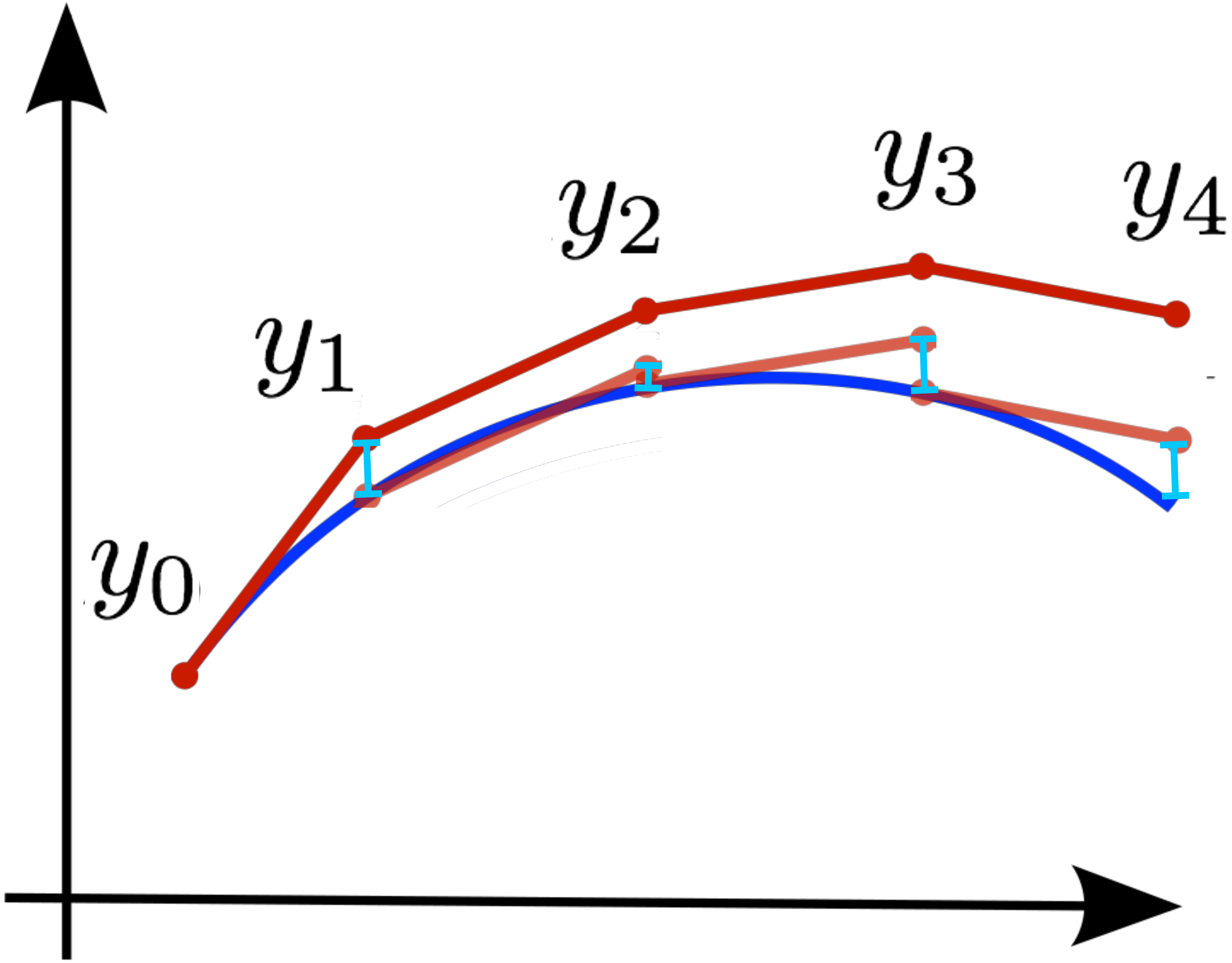
Add to current state
$y_0 + f(t_0, y_0)dt = y_1$
y_0
Euler integration
y_0
y_1
y_2
y_3
y_4
True evolution of state

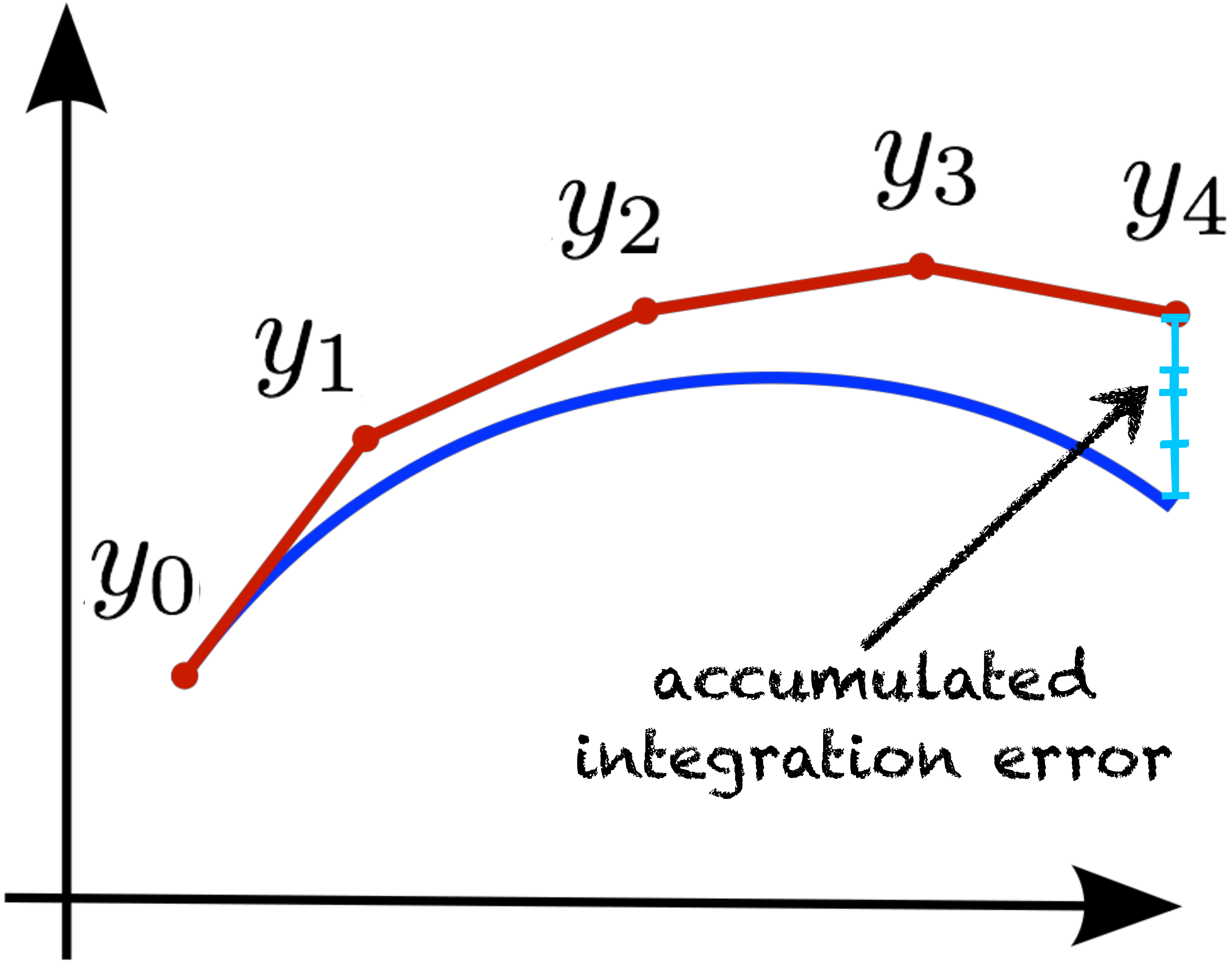


$y_0 + f(t_0, y_0)dt = y_1$
y_0
Integration error
Euler integration
y_1
y_2
y_3
y_4
y_0
True evolution of state



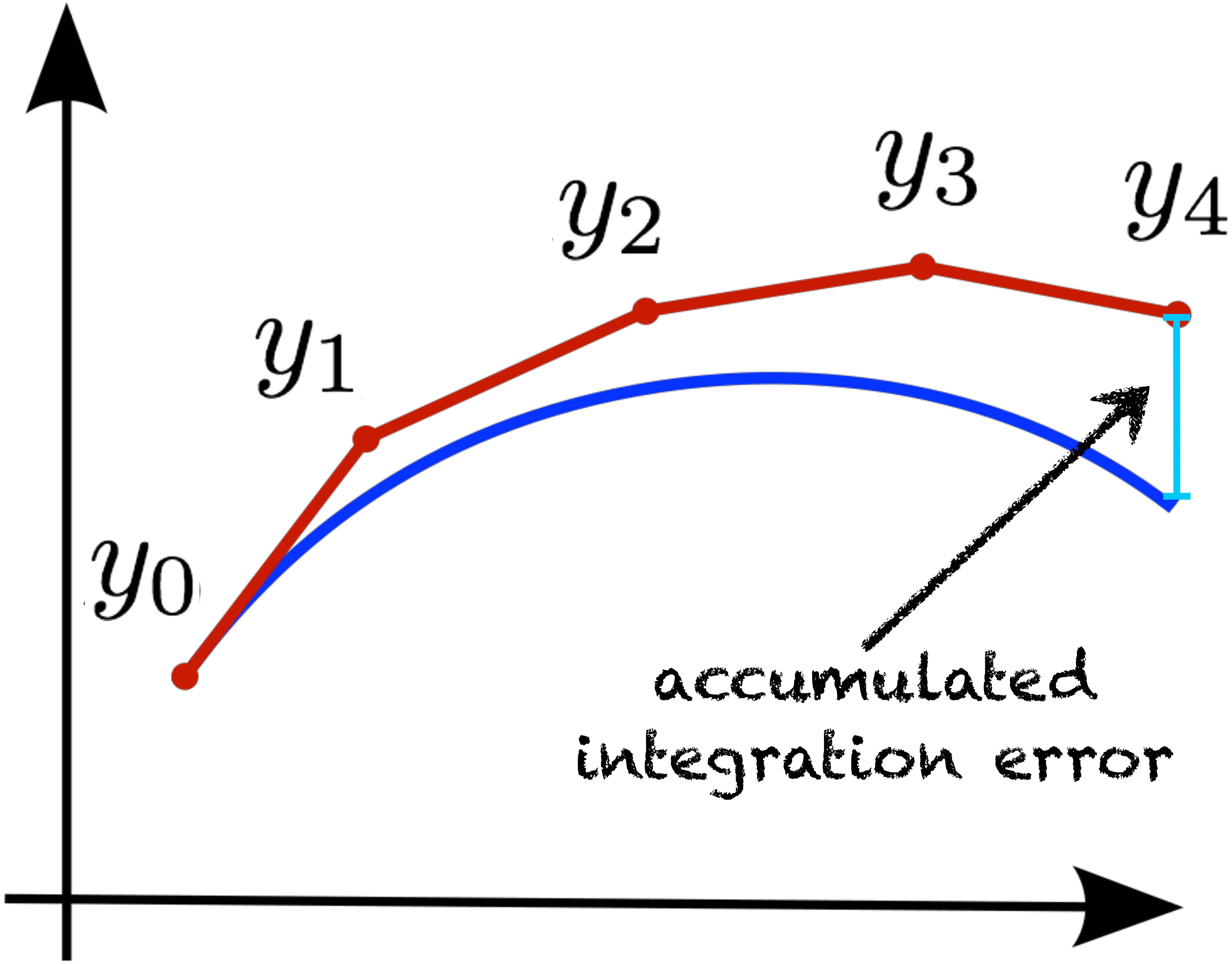
Euler integration
$y_0 + f(t_0, y_0)dt = y_1$
y_0
y_1
y_2
y_3
y_4
True evolution of state
Integration error


y_0
y_1
y_2
y_3
y_4



y_0
y_1
y_2
y_3
y_4
accumulated integration error

Can we improve this integration over time?



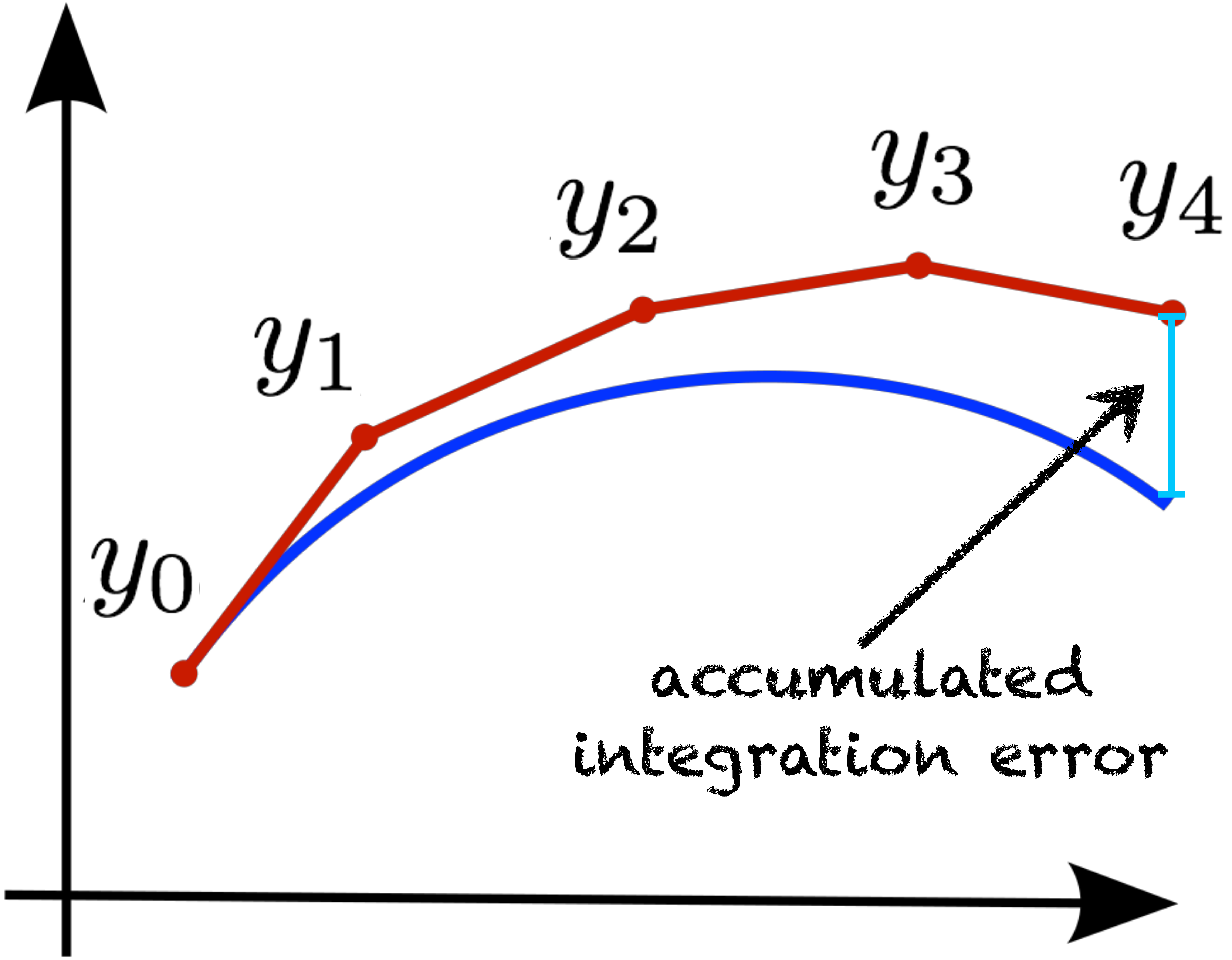
Can we improve this integration over time?

Option 1:

[redacted]

Option 2:

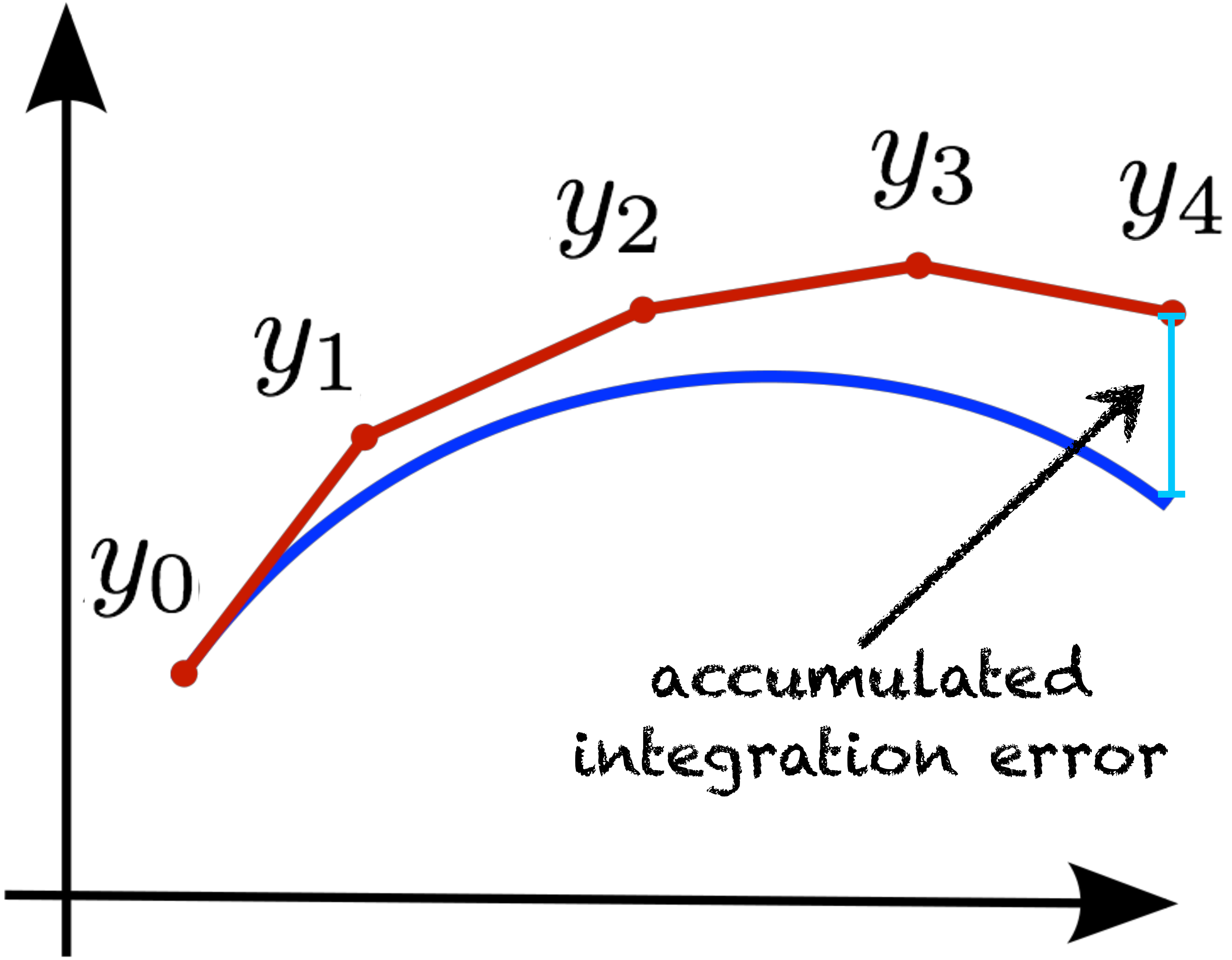
[redacted]



Can we improve this integration over time?

Option 1:
Reduce timestep

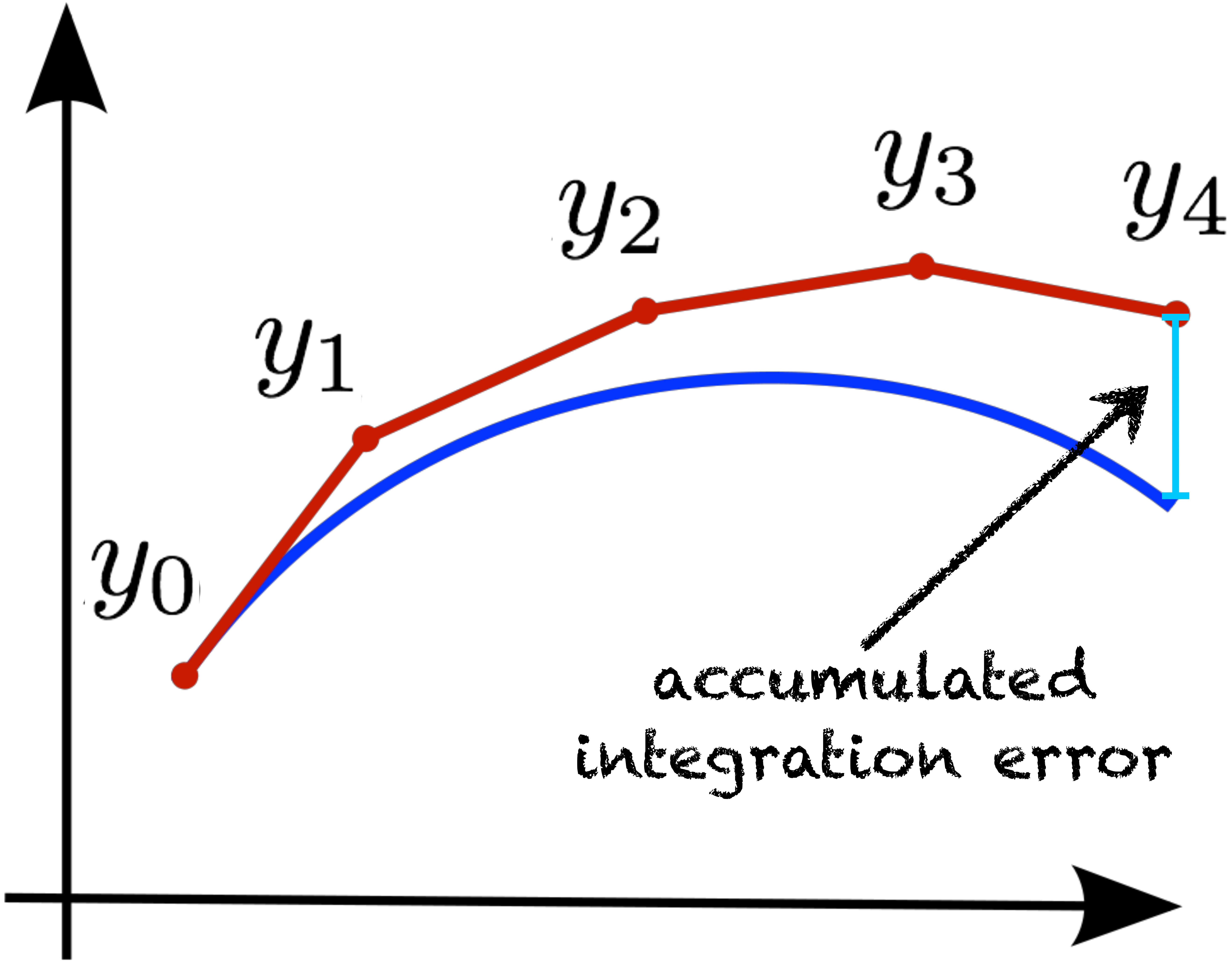
Option 2:
[redacted]



Can we improve this integration over time?

Option 1:
Reduce timestep

Option 2:
Use a better integrator



Verlet Integration

Verlet integration

For a differential equation of second order of the type $\ddot{\vec{x}}(t) = A(\vec{x}(t))$ with initial conditions $\vec{x}(t_0) = \vec{x}_0$ and $\dot{\vec{x}}(t_0) = \vec{v}_0$, an approximate numerical solution $\vec{x}_n \approx \vec{x}(t_n)$ at the times $t_n = t_0 + n\,\Delta t$ with step size $\Delta t > 0$ can be obtained by the following method:

Advance position $$y_{n+1} = 2y_n - y_{n-1} + a(y_n)h^2$$

Verlet Integration

$$a(y_n) = \frac{\Delta^2 y_n}{\Delta t^2}$$

Start with discrete time approximation of acceleration at time n

Verlet Integration

$$a(y_n) = \frac{\Delta^2 y_n}{\Delta t^2}$$

Start with discrete time approximation of acceleration at time n

$$\approx \frac{\dot{y}_n - \dot{y}_{n-1}}{\Delta t}$$

breakdown into velocities

Verlet Integration

$$a(y_n) = \frac{\Delta^2 y_n}{\Delta t^2}$$

Start with discrete time approximation of acceleration at time n

$$\approx \frac{\dot{y}_n - \dot{y}_{n-1}}{\Delta t}$$

breakdown into velocities

$$\approx \frac{\frac{y_{n+1} - y_n}{\Delta t} - \frac{y_n - y_{n-1}}{\Delta t}}{\Delta t}$$

and then positions

Verlet Integration

$$a(y_n) = \frac{\Delta^2 y_n}{\Delta t^2}$$

Start with discrete time approximation of acceleration at time n

$$\approx \frac{\dot{y}_n - \dot{y}_{n-1}}{\Delta t}$$

breakdown into velocities

$$\approx \frac{\frac{y_{n+1}-y_n}{\Delta t} - \frac{y_n - y_{n-1}}{\Delta t}}{\Delta t}$$

and then positions

$$= \frac{y_{n+1} - 2y_n + y_{n-1}}{\Delta t^2}$$

do some algebra

Verlet Integration

$$a(y_n) = \frac{\Delta^2 y_n}{\Delta t^2}$$

Start with discrete time approximation of acceleration at time n

$$\approx \frac{\dot{y}_n - \dot{y}_{n-1}}{\Delta t}$$

breakdown into velocities

$$\approx \frac{\frac{y_{n+1} - y_n}{\Delta t} - \frac{y_n - y_{n-1}}{\Delta t}}{\Delta t}$$

and then positions

$$= \frac{y_{n+1} - 2y_n + y_{n-1}}{\Delta t^2}$$

do some algebra

solve for next state

$$\Rightarrow \boxed{y_{n+1} \approx 2y_n - y_{n-1} + a(y_n)\Delta t^2}$$

Verlet Integration

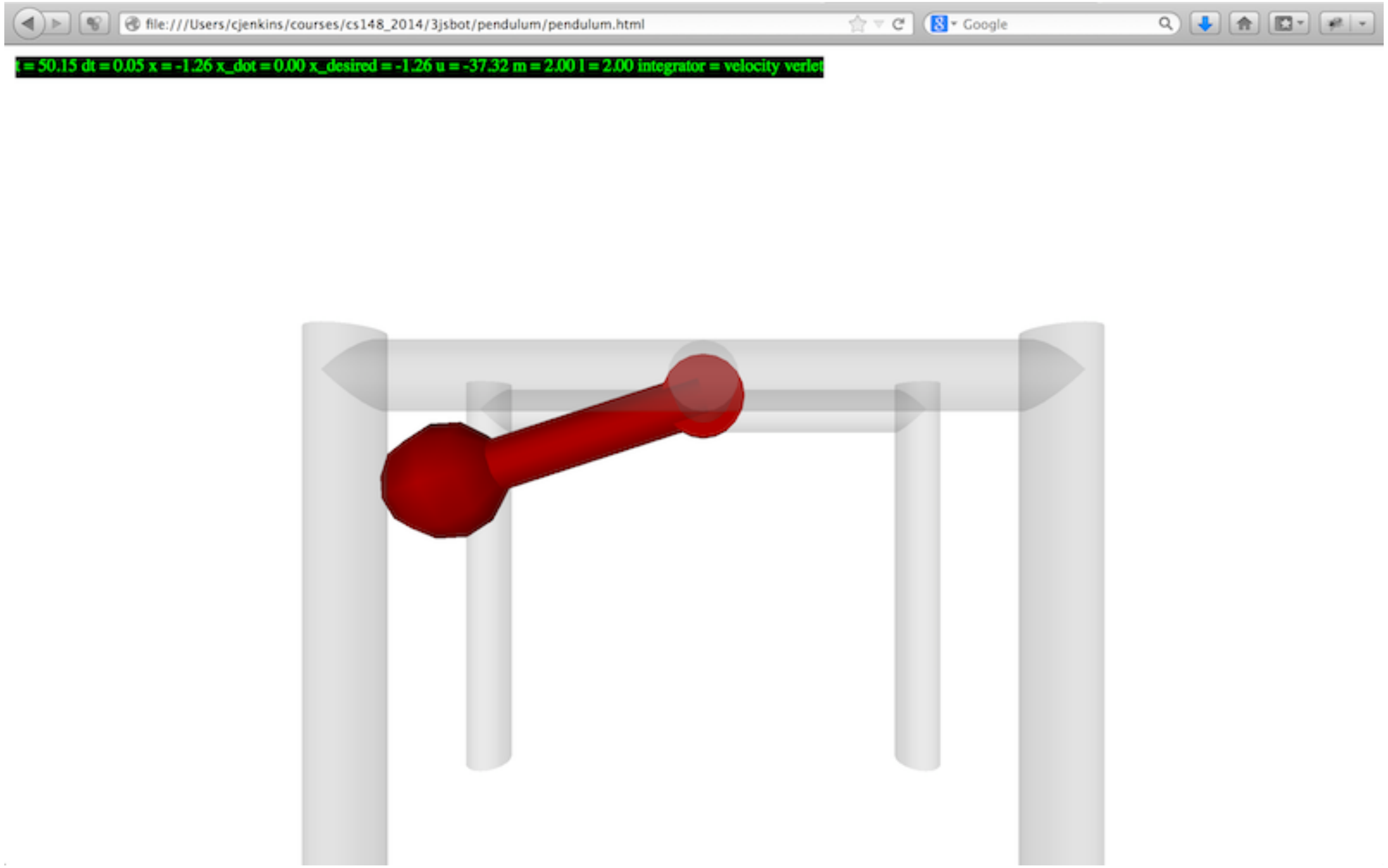
For a differential equation of second order of the type $\ddot{\vec{x}}(t) = A(\vec{x}(t))$ with initial conditions $\vec{x}(t_0) = \vec{x}_0$ and $\dot{\vec{x}}(t_0) = \vec{v}_0$, an approximate numerical solution $\vec{x}_n \approx \vec{x}(t_n)$ at the times $t_n = t_0 + n\,\Delta t$ with step size $\Delta t > 0$ can be obtained by the following method:

Initialize $$y_1 = y_0 + h\dot{y}_0 + h^2\frac{1}{2}a(y_0)$$

Advance position $$y_{n+1} = 2y_n - y_{n-1} + a(y_n)h^2$$

Do not forget to initialize

Let’s see what happens



How does Verlet integrate velocity?

Verlet Integration

For a differential equation of second order of the type $\ddot{\vec{x}}(t) = A(\vec{x}(t))$ with initial conditions $\vec{x}(t_0) = \vec{x}_0$ and $\dot{\vec{x}}(t_0) = \vec{v}_0$, an approximate numerical solution $\vec{x}_n \approx \vec{x}(t_n)$ at the times $t_n = t_0 + n\,\Delta t$ with step size $\Delta t > 0$ can be obtained by the following method:

Initialize $$y_1 = y_0 + h\dot{y}_0 + h^2\frac{1}{2}a(y_0)$$

Advance position $$y_{n+1} = 2y_n - y_{n-1} + a(y_n)h^2$$

Advance velocity (optional) $$\dot{y}_n = \frac{y(t+h) - y(t-h)}{2h} + \mathcal{O}(h^2)$$

Verlet Integration

For a differential equation of second order of the type $\ddot{\vec{x}}(t) = A(\vec{x}(t))$ with initial conditions $\vec{x}(t_0) = \vec{x}_0$ and $\dot{\vec{x}}(t_0) = \vec{v}_0$, an approximate numerical solution $\vec{x}_n \approx \vec{x}(t_n)$ at the times $t_n = t_0 + n\,\Delta t$ with step size $\Delta t > 0$ can be obtained by the following method:

is there a cleaner alternative?

Initialize $$y_1 = y_0 + h\dot{y}_0 + h^2\frac{1}{2}a(y_0)$$

Advance position $$y_{n+1} = 2y_n - y_{n-1} + a(y_n)h^2$$

Advance velocity (optional) $$\dot{y}_n = \frac{y(t+h) - y(t-h)}{2h} + \mathcal{O}(h^2)$$

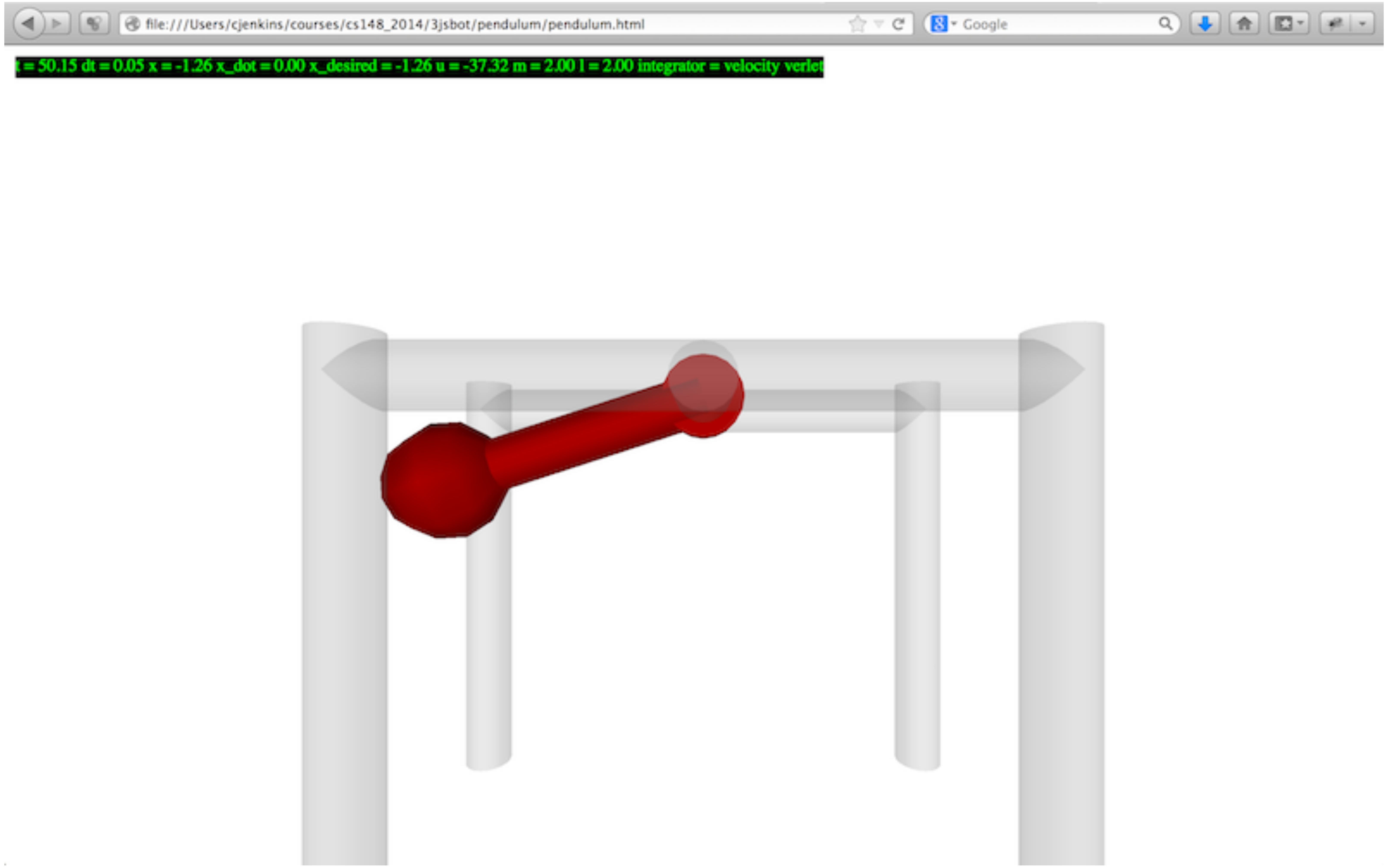
Velocity Verlet

$$y(t+\Delta t) = y(t) + \dot{y}(t)\Delta t + \frac{1}{2}a(t)\Delta t^2$$

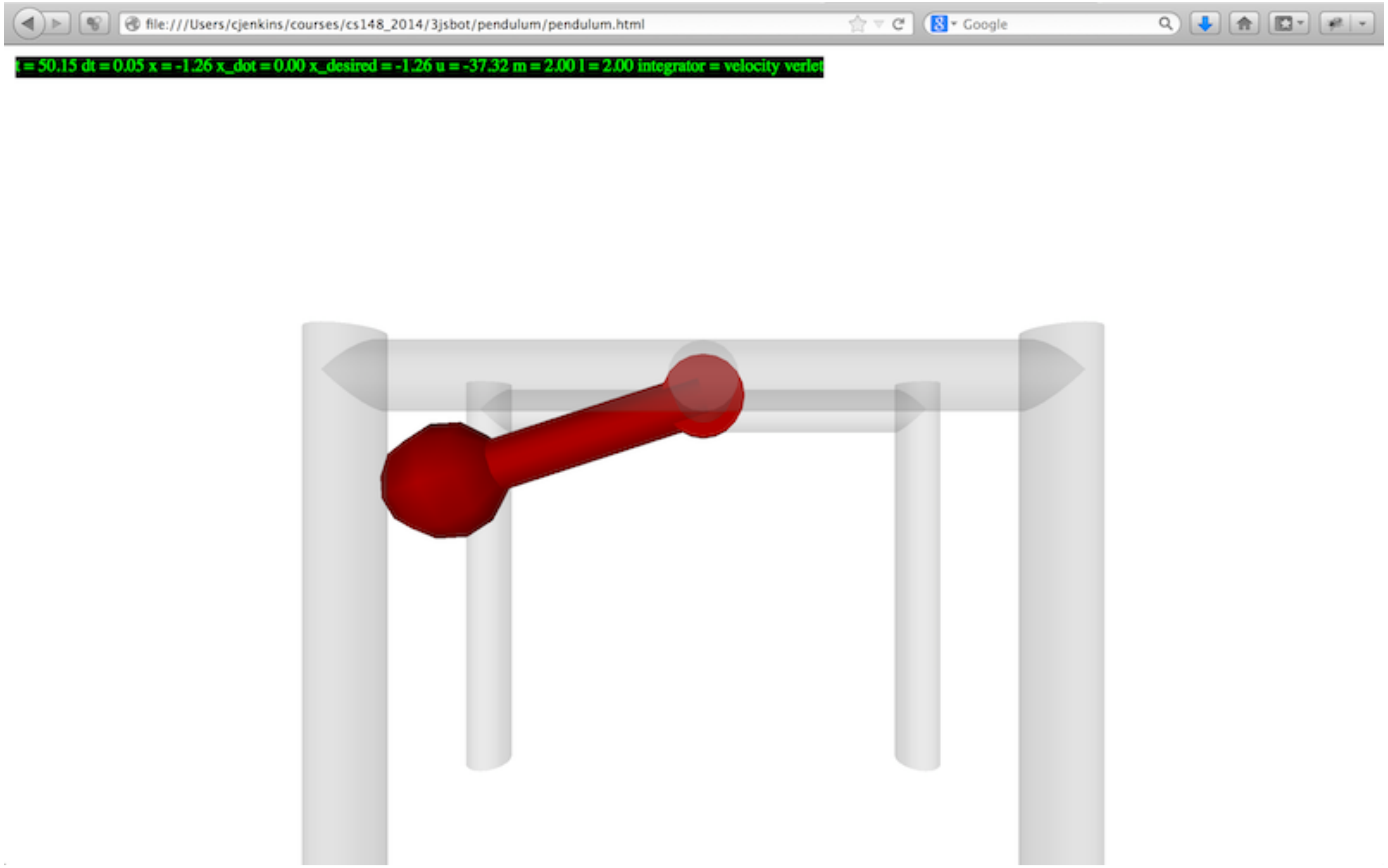
$$\dot{y}(t+\Delta t) = \dot{y}(t) + \frac{a(t) + a(t+\Delta t)}{2}\Delta t$$

assumes that acceleration $a(t+\Delta t)$
only depends on position $y(t+\Delta t)$

Let's see what happens

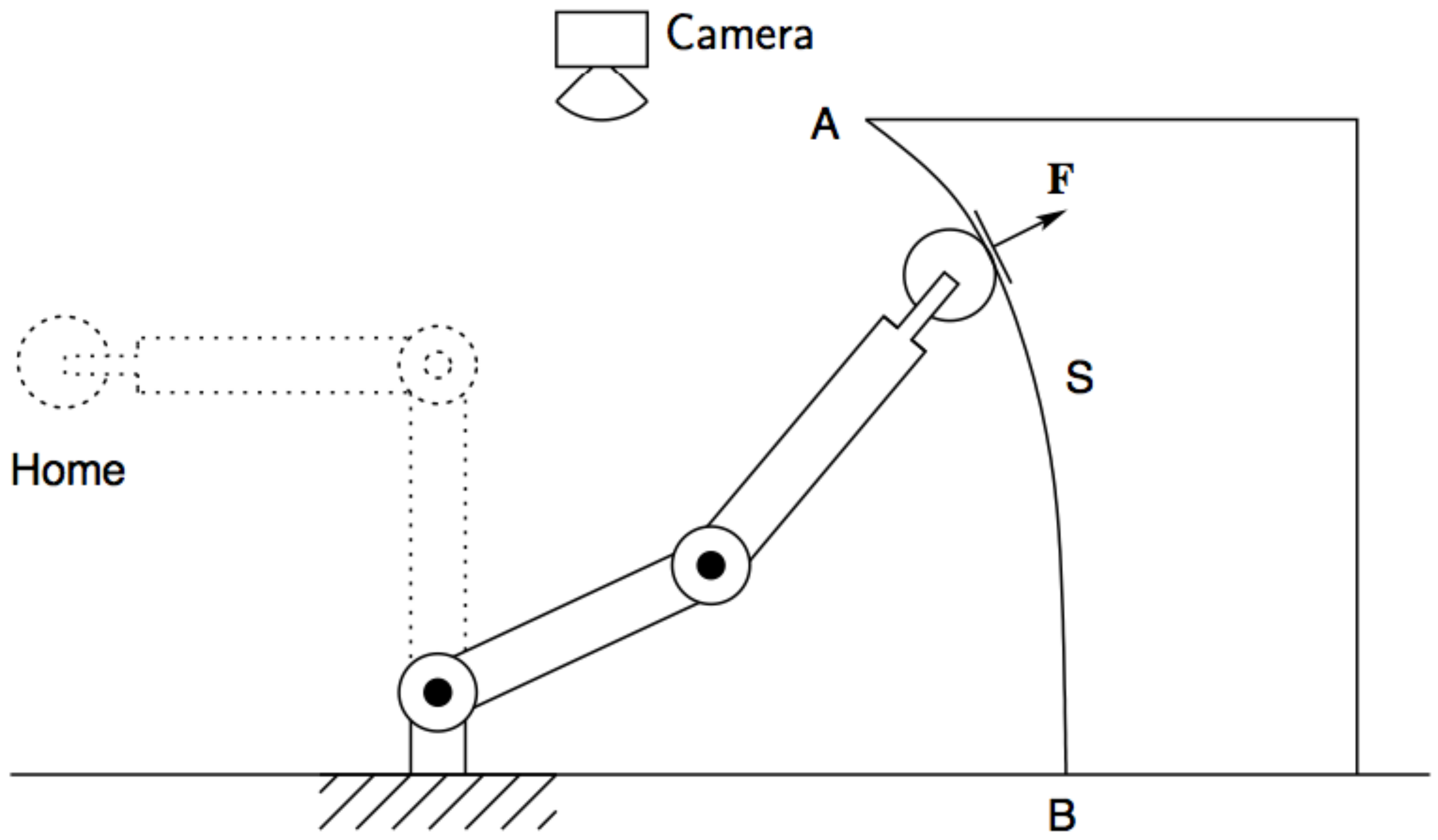


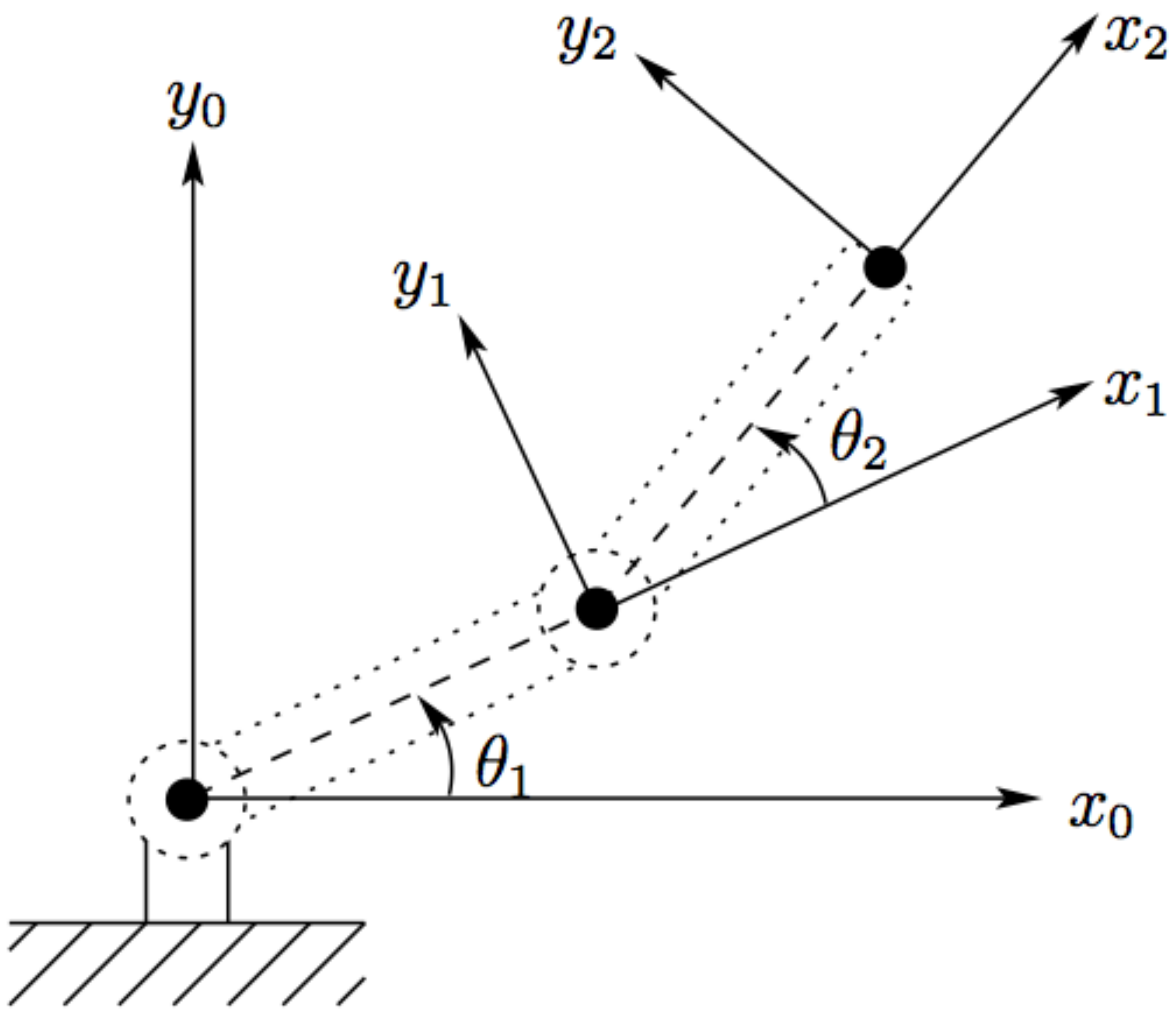
Let’s see what happens



Let’s revisit the Planar 2-DOF 2-link Arm

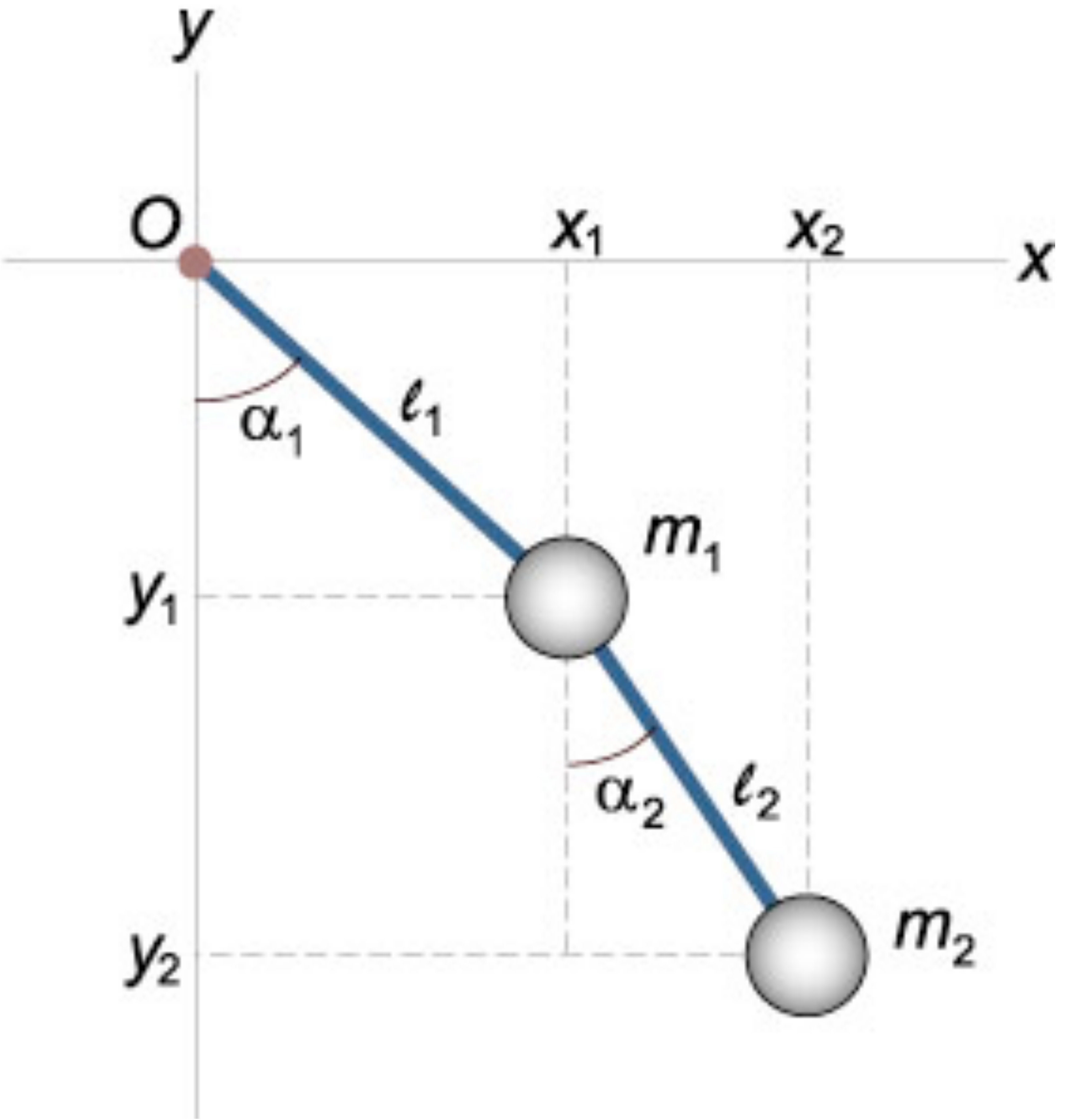
Planar 2-DOF 2-link Arm



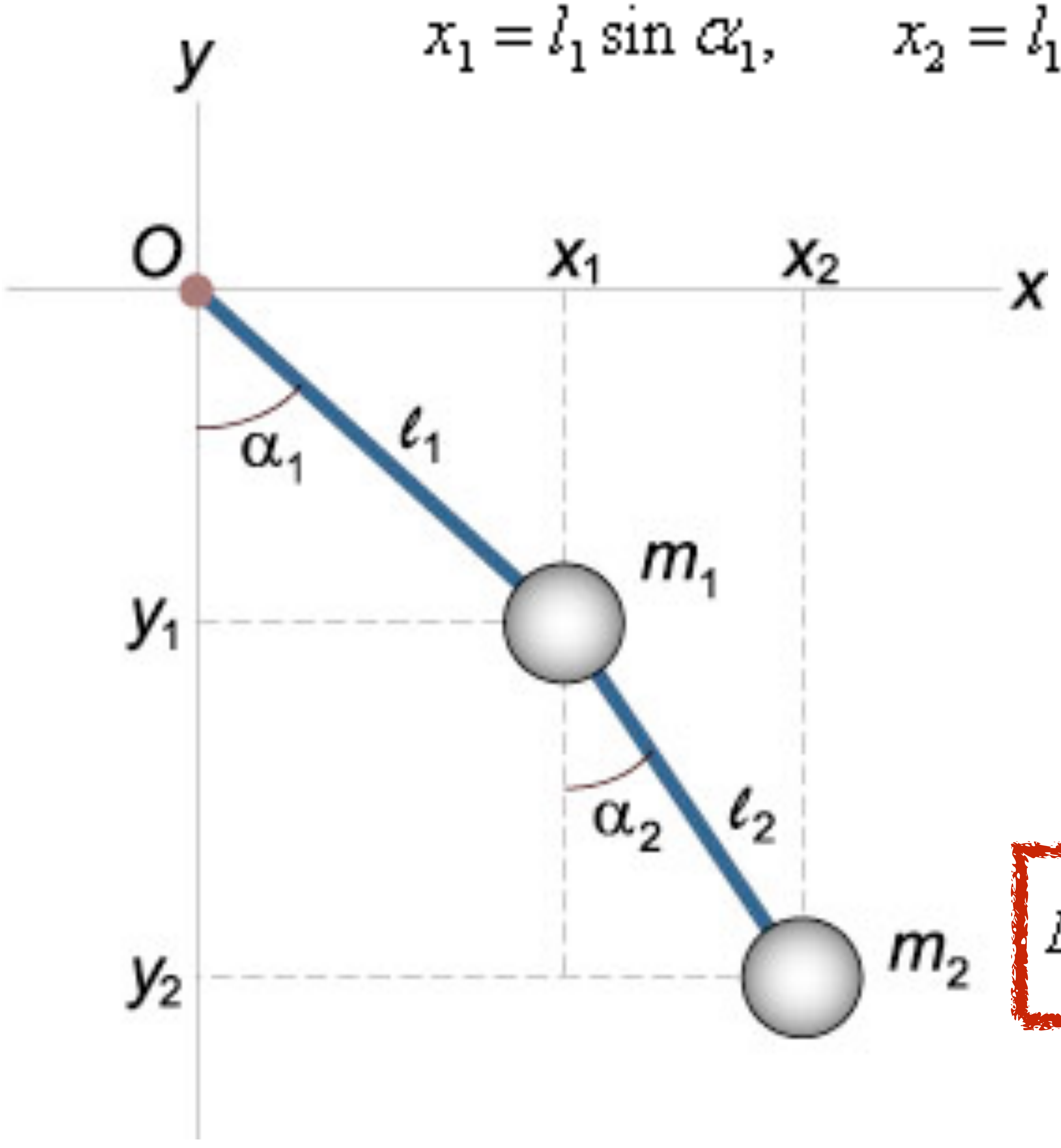



For Coffee Supplies
Please See Yolonda in
Room 3820
Domino

Can we add another link?



- Double pendulum



Locations of pendulum bobs

$$x_1 = l_1 \sin\alpha_1, \quad x_2 = l_1 \sin\alpha_1 + l_2 \sin\alpha_2, \quad y_1 = -l_1 \cos\alpha_1, \quad y_2 = -l_1 \cos\alpha_1 - l_2 \cos\alpha_2.$$

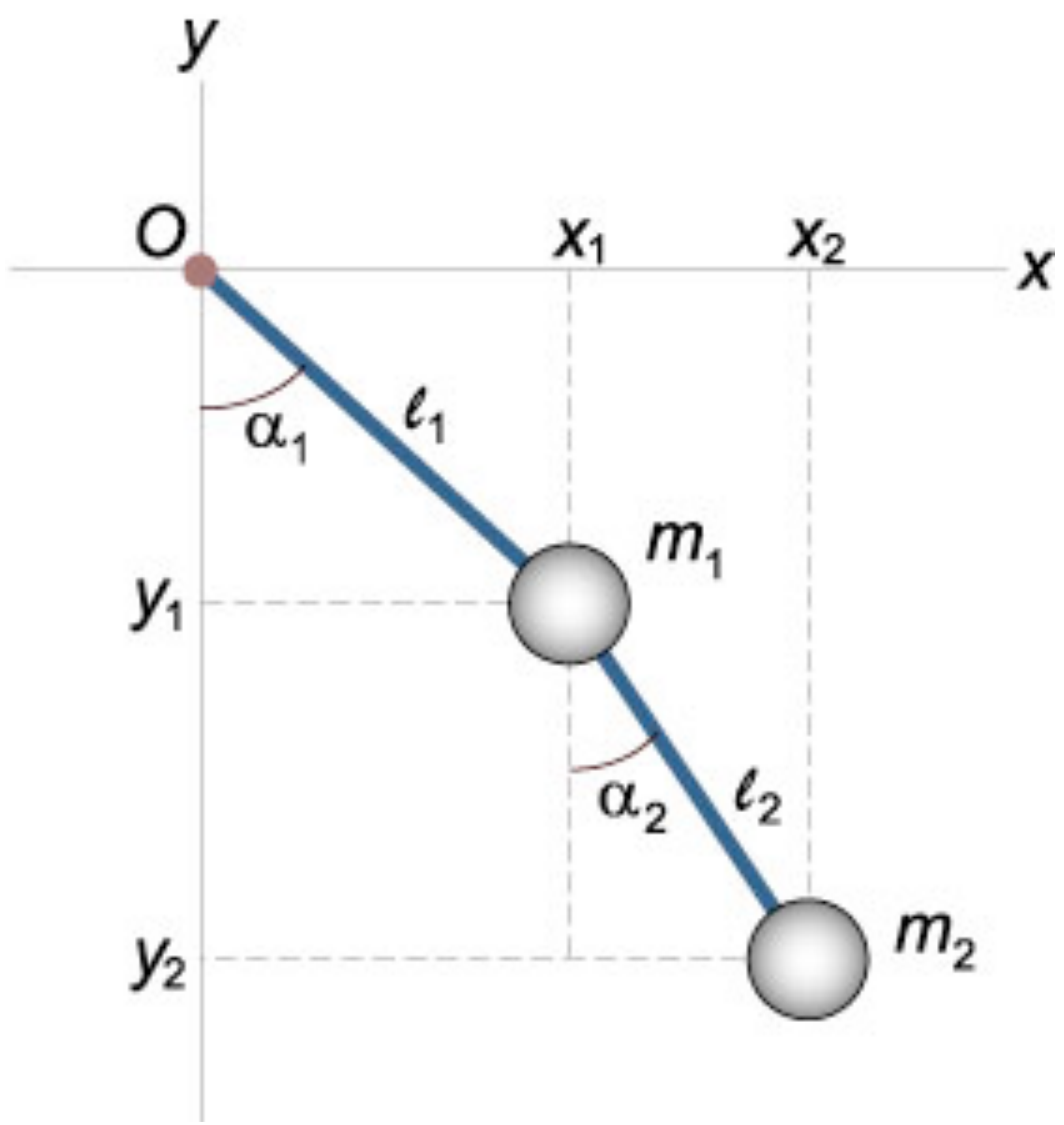
Lagrangian of pendulum bob positions

$$T = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} = \frac{m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right)}{2} + \frac{m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right)}{2}, \qquad V = m_1 g y_1 + m_2 g y_2.$$

$$\boxed{L = T - V = T_1 + T_2 - (V_1 + V_2) = \frac{m_1}{2}\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{m_2}{2}\left(\dot{x}_2^2 + \dot{y}_2^2\right) - m_1 g y_1 - m_2 g y_2.}$$

http://www.math24.net/double-pendulum.html

Lagrangian in generalized coordinates (joint angles)



http://www.math24.net/double-pendulum.html

$$\dot{x}_1 = l_1 \cos\alpha_1 \cdot \dot{\alpha}_1, \qquad \dot{x}_2 = l_1 \cos\alpha_1 \cdot \dot{\alpha}_1 + l_2 \cos\alpha_2 \cdot \dot{\alpha}_2,$$

$$\dot{y}_1 = l_1 \sin\alpha_1 \cdot \dot{\alpha}_1, \qquad \dot{y}_2 = l_1 \sin\alpha_1 \cdot \dot{\alpha}_1 + l_2 \sin\alpha_2 \cdot \dot{\alpha}_2.$$

$$T_1 = \frac{m_1}{2}\left(\dot{x}_1^2 + \dot{y}_1^2\right) = \frac{m_1}{2}\left(l_1^2\dot{\alpha}_1^2\cos^2\alpha_1 + l_1^2\dot{\alpha}_1^2\sin^2\alpha_1\right) = \frac{m_1}{2}l_1^2\dot{\alpha}_1^2,$$

$$T_2 = \frac{m_2}{2}\left(\dot{x}_2^2 + \dot{y}_2^2\right) = \frac{m_2}{2}\left[\left(l_1\dot{\alpha}_1\cos\alpha_1 + l_2\dot{\alpha}_2\cos\alpha_2\right)^2 + \left(l_1\dot{\alpha}_1\sin\alpha_1 + l_2\dot{\alpha}_2\sin\alpha_2\right)^2\right]$$

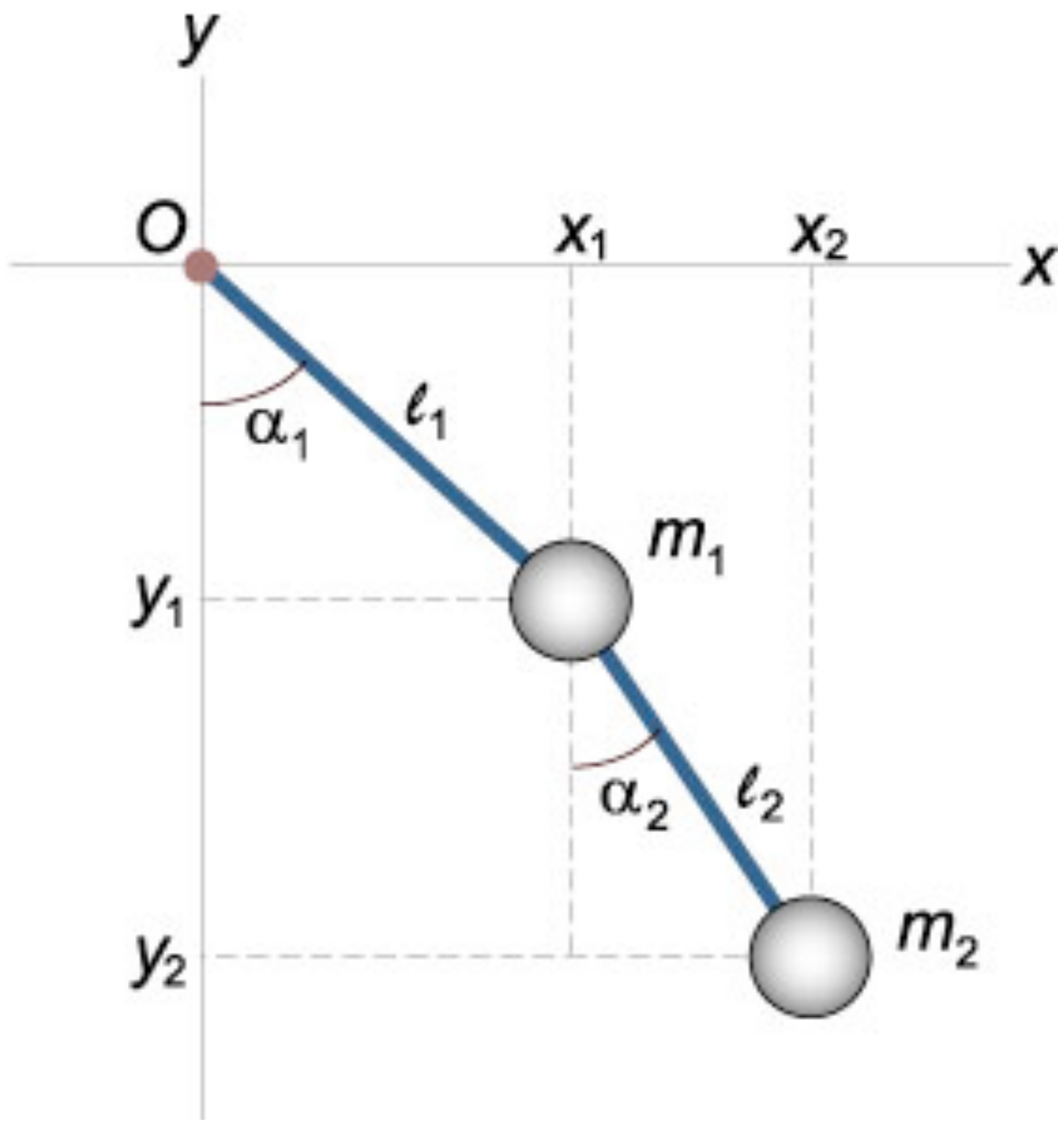
$$= \frac{m_2}{2}\left[l_1^2\dot{\alpha}_1^2\cos^2\alpha_1 + l_2^2\dot{\alpha}_2^2\cos^2\alpha_2 + 2l_1l_2\dot{\alpha}_1\dot{\alpha}_2\cos\alpha_1\cos\alpha_2 + l_1^2\dot{\alpha}_1^2\sin^2\alpha_1 + l_2^2\dot{\alpha}_2^2\sin^2\alpha_2 + 2l_1l_2\dot{\alpha}_1\dot{\alpha}_2\sin\alpha_1\sin\alpha_2\right]$$

$$= \frac{m_2}{2}\left[l_1^2\dot{\alpha}_1^2 + l_2^2\dot{\alpha}_2^2 + 2l_1l_2\dot{\alpha}_1\dot{\alpha}_2\cos\left(\alpha_1 - \alpha_2\right)\right],$$

$$V_1 = m_1 g y_1 = -m_1 g l_1 \cos\alpha_1,$$

$$V_2 = m_2 g y_2 = -m_2 g\left(l_1\cos\alpha_1 + l_2\cos\alpha_2\right).$$

$$\boxed{\begin{aligned} L &= T - V = T_1 + T_2 - (V_1 + V_2) = \\ &= \left(\frac{m_1}{2} + \frac{m_2}{2}\right)l_1^2\dot{\alpha}_1^2 + \frac{m_2}{2}l_2^2\dot{\alpha}_2^2 + m_2 l_1 l_2 \dot{\alpha}_1\dot{\alpha}_2\cos\left(\alpha_1 - \alpha_2\right) + \left(m_1 + m_2\right)g l_1\cos\alpha_1 + m_2 g l_2\cos\alpha_2.\end{aligned}}$$



http://www.math24.net/double-pendulum.html

Lagrangian equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}_i} - \frac{\partial L}{\partial \alpha_i} = 0, \qquad i = 1,2.$$

$$\frac{\partial L}{\partial \dot{\alpha}_1} = (m_1+m_2)\,l_1^2\dot{\alpha}_1 + m_2 l_1 l_2 \dot{\alpha}_2 \cos(\alpha_1-\alpha_2), \qquad \frac{\partial L}{\partial \alpha_1} = -m_2 l_1 l_2 \dot{\alpha}_1\dot{\alpha}_2 \sin(\alpha_1-\alpha_2) - (m_1+m_2)\,g l_1 \sin\alpha_1,$$

$$\frac{\partial L}{\partial \dot{\alpha}_2} = m_2 l_2^2 \dot{\alpha}_2 + m_2 l_1 l_2 \dot{\alpha}_1 \cos(\alpha_1-\alpha_2), \qquad \frac{\partial L}{\partial \alpha_2} = m_2 l_1 l_2 \dot{\alpha}_1\dot{\alpha}_2 \sin(\alpha_1-\alpha_2) - m_2 g l_2 \sin\alpha_2.$$

Lagrangian EOM for first DOF (*i*=*1*)

$$\frac{d}{dt}\left[(m_1+m_2)\,l_1^2\dot{\alpha}_1 + m_2 l_1 l_2 \dot{\alpha}_2 \cos(\alpha_1-\alpha_2)\right] + m_2 l_1 l_2 \dot{\alpha}_1\dot{\alpha}_2 \sin(\alpha_1-\alpha_2) + (m_1+m_2)\,g l_1 \sin\alpha_1 = 0,$$

$$\Rightarrow (m_1+m_2)\,l_1^2\ddot{\alpha}_1 + m_2 l_1 l_2\left[\ddot{\alpha}_2 \cos(\alpha_1-\alpha_2) - \dot{\alpha}_2 \sin(\alpha_1-\alpha_2)\cdot(\dot{\alpha}_1-\dot{\alpha}_2)\right]$$

$$+ m_2 l_1 l_2 \dot{\alpha}_1\dot{\alpha}_2 \sin(\alpha_1-\alpha_2) + (m_1+m_2)\,g l_1 \sin\alpha_1 = 0,$$

$$\Rightarrow (m_1+m_2)\,l_1^2\ddot{\alpha}_1 + m_2 l_1 l_2 \ddot{\alpha}_2 \cos(\alpha_1-\alpha_2) - \cancel{m_2 l_1 l_2 \dot{\alpha}_1\dot{\alpha}_2 \sin(\alpha_1-\alpha_2)} + m_2 l_1 l_2 \dot{\alpha}_2^2 \sin(\alpha_1-\alpha_2)$$

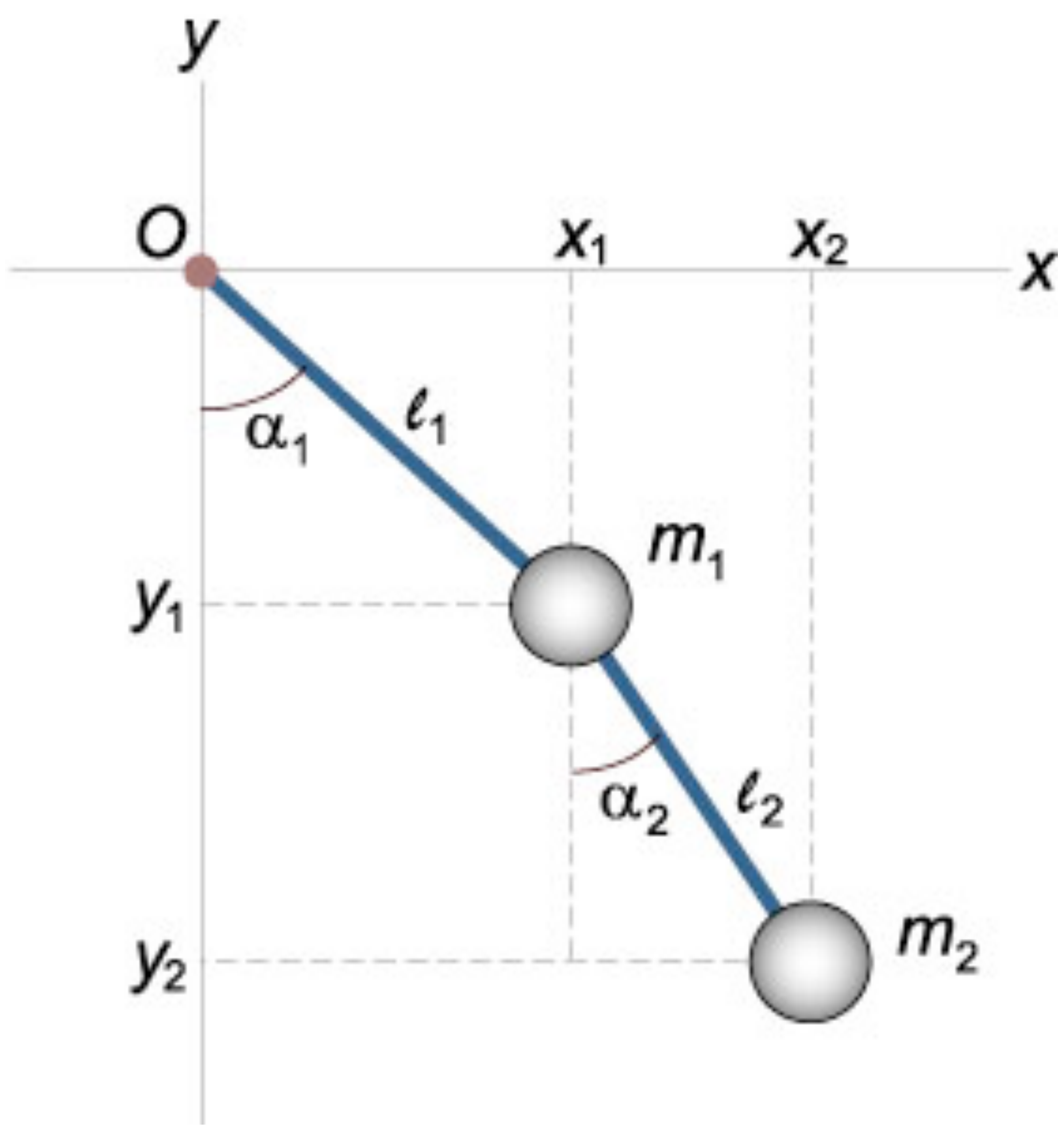
$$+ \cancel{m_2 l_1 l_2 \dot{\alpha}_1\dot{\alpha}_2 \sin(\alpha_1-\alpha_2)} + (m_1+m_2)\,g l_1 \sin\alpha_1 = 0,$$

$$\Rightarrow \boxed{(m_1+m_2)\,l_1^2\ddot{\alpha}_1 + m_2 l_1 l_2 \ddot{\alpha}_2 \cos(\alpha_1-\alpha_2) + m_2 l_1 l_2 \dot{\alpha}_2^2 \sin(\alpha_1-\alpha_2) + (m_1+m_2)\,g l_1 \sin\alpha_1 = 0.}$$

$$(m_1+m_2)\,l_1\ddot{\alpha}_1 + m_2 l_2 \ddot{\alpha}_2 \cos(\alpha_1-\alpha_2) + m_2 l_2 \dot{\alpha}_2^2 \sin(\alpha_1-\alpha_2) + (m_1+m_2)\,g \sin\alpha_1 = 0.$$

Lagrangian equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}_i} - \frac{\partial L}{\partial \alpha_i} = 0, \qquad i = 1,2.$$



$$\frac{\partial L}{\partial \dot{\alpha}_1} = (m_1+m_2) l_1^2 \dot{\alpha}_1 + m_2 l_1 l_2 \dot{\alpha}_2 \cos(\alpha_1 - \alpha_2), \qquad \frac{\partial L}{\partial \alpha_1} = -m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2) - (m_1+m_2) g l_1 \sin \alpha_1,$$

$$\frac{\partial L}{\partial \dot{\alpha}_2} = m_2 l_2^2 \dot{\alpha}_2 + m_2 l_1 l_2 \dot{\alpha}_1 \cos(\alpha_1 - \alpha_2), \qquad \frac{\partial L}{\partial \alpha_2} = m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2) - m_2 g l_2 \sin \alpha_2.$$

Lagrangian EOM for second DOF (*i*=2)

$$\frac{d}{dt}\left[m_2 l_2^2 \dot{\alpha}_2 + m_2 l_1 l_2 \dot{\alpha}_1 \cos(\alpha_1 - \alpha_2)\right] - m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2) + m_2 g l_2 \sin \alpha_2 = 0,$$

$$\Rightarrow m_2 l_2^2 \ddot{\alpha}_2 + m_2 l_1 l_2 \ddot{\alpha}_1 \cos(\alpha_1 - \alpha_2) - m_2 l_1 l_2 \dot{\alpha}_1 \sin(\alpha_1 - \alpha_2) \cdot (\dot{\alpha}_1 - \dot{\alpha}_2) - m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2)$$

$$+ m_2 g l_2 \sin \alpha_2 = 0,$$

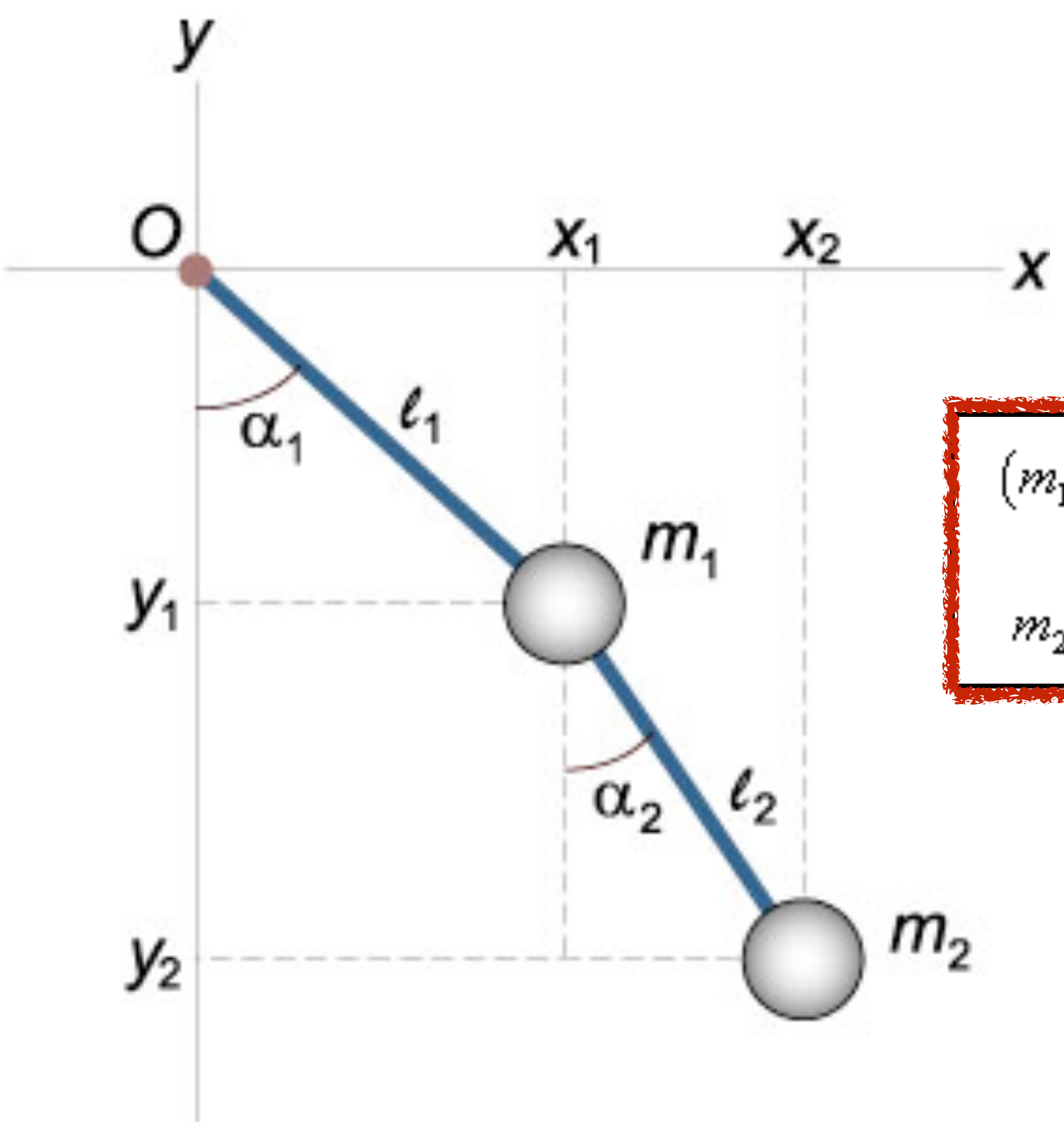
$$\Rightarrow m_2 l_2^2 \ddot{\alpha}_2 + m_2 l_1 l_2 \ddot{\alpha}_1 \cos(\alpha_1 - \alpha_2) - m_2 l_1 l_2 \dot{\alpha}_1^2 \sin(\alpha_1 - \alpha_2) + \cancel{m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2)}$$

$$- \cancel{m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2)} + m_2 g l_2 \sin \alpha_2 = 0,$$

$$\Rightarrow \boxed{m_2 l_2^2 \ddot{\alpha}_2 + m_2 l_1 l_2 \ddot{\alpha}_1 \cos(\alpha_1 - \alpha_2) - m_2 l_1 l_2 \dot{\alpha}_1^2 \sin(\alpha_1 - \alpha_2) + m_2 g l_2 \sin \alpha_2 = 0.}$$

$$l_2 \ddot{\alpha}_2 + l_1 \ddot{\alpha}_1 \cos(\alpha_1 - \alpha_2) - l_1 \dot{\alpha}_1^2 \sin(\alpha_1 - \alpha_2) + g \sin \alpha_2 = 0.$$

http://www.math24.net/double-pendulum.html



http://www.math24.net/double-pendulum.html

The 2 equations of motion for double pendulum

$$(m_1+m_2)l_1^2\ddot{\alpha}_1+m_2l_1l_2\ddot{\alpha}_2\cos(\alpha_1-\alpha_2)+m_2l_1l_2\dot{\alpha}_2^2\sin(\alpha_1-\alpha_2)+(m_1+m_2)gl_1\sin\alpha_1=0$$

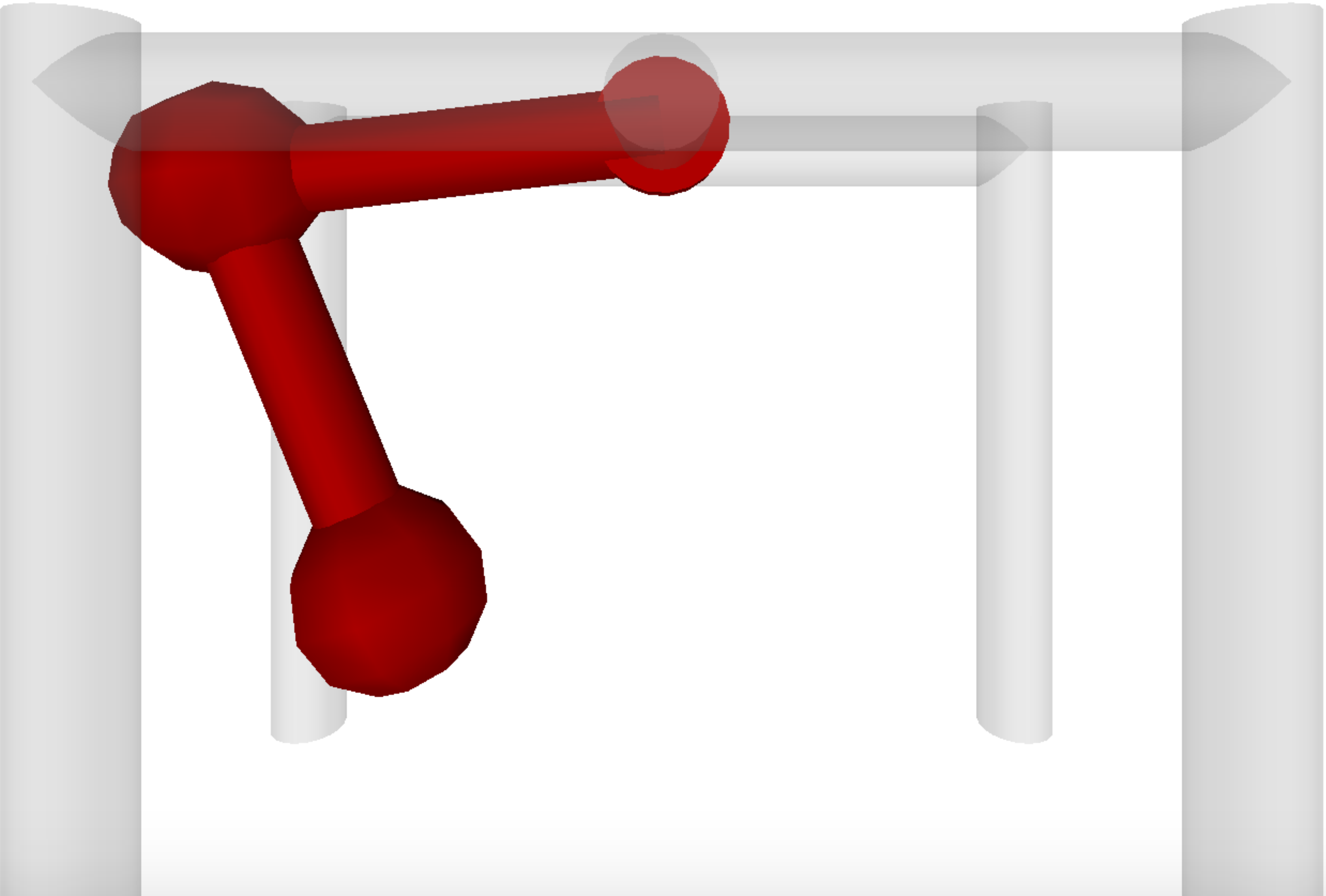
$$m_2l_2^2\ddot{\alpha}_2+m_2l_1l_2\ddot{\alpha}_1\cos(\alpha_1-\alpha_2)-m_2l_1l_2\dot{\alpha}_1^2\sin(\alpha_1-\alpha_2)+m_2gl_2\sin\alpha_2=0$$

Substitute equations into each other to solve for $\ddot{\alpha}_1$ and $\ddot{\alpha}_2$

What happened to the torques?

Pendularm2 by oharib

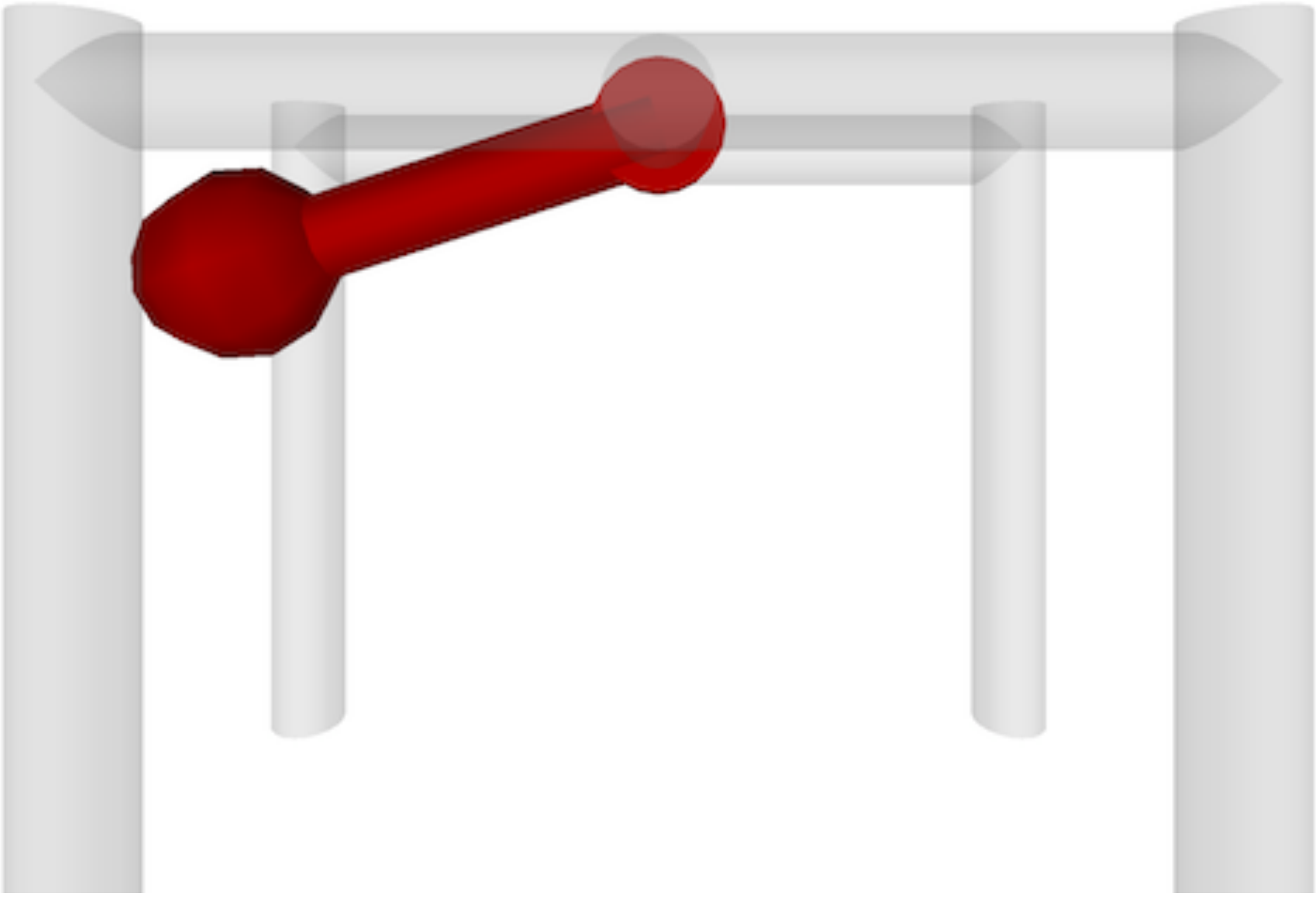
```
System
t = 6119.20 dt = 0.05
integrator = runge-kutta
x1 = -1.46
x1_dot = -0.00
x2 = 1.84
x2_dot = -0.00
x1_desired = -1.46
x2_desired = 1.84
```



```
Pendulum
mass = 2.00
length = 2.00
gravity = 9.81

Keys
a/d - apply user force
1/2 - adjust desired angle 1
3/4 - adjust desired angle 2
c|x - toggle servo
s - disable servo
```

Project 2: Pendularm



https://raw.githubusercontent.com/autorob/kineval-stencil/master/project_pendularm/pendularm1.html

```
function init() {...
    pendulum = { // pendulum object
        length:2.0,
        mass:2.0,
        angle:Math.PI/2,
        angle_dot:0.0};

    gravity = 9.81;  // Earth gravity
    t = 0; dt = 0.05; // init time
    pendulum.control = 0; // motor

    // next lecture: PID control
    pendulum.desired = -Math.PI/2.5;
    pendulum.desired_dot = 0;
    pendulum.servo =
        {kp:0, kd:0, ki:0};
...}
```

```
function animate() { ...
if (numerical_integrator === "euler") {
    // STENCIL: Euler integrator }
    else if (numerical_integrator === "verlet") {
    // STENCIL: basic Verlet integration }
    else if (numerical_integrator === "velocity verlet") {
    // STENCIL: velocity Verlet }
    else if (numerical_integrator === "runge-kutta") {
    // STENCIL: Runge-Kutta 4 integrator }
    else {   }
    // set the angle of the pendulum
    pendulum.geom.rotation.y = pendulum.angle;
    t = t + dt;  // advance time
...}

function pendulum_acceleration(p,g) {
   // STENCIL: return acceleration from equations of motion }
```


Next up:
Control
(although no Janet Jackson)